Lecture 24: Randomized Algorithms

Chapter 12
Study Session tomorrow
Virtual 4th problem set,
Return PS#1, PS#2
Randomized Algorithms

• Randomized algorithms incorporate random, rather than deterministic, decisions
• Commonly used in situations where no exact and/or fast algorithm is known
• Main advantage is that no input can reliably produce worst-case results because the algorithm runs differently each time.
Select

- \textbf{Select}(L, k) \text{ finds the } k^{\text{th}} \text{ smallest element in } L
- \text{Select}(L, 1) \text{ find the smallest…}
  - Well known O(n) algorithm
    \begin{verbatim}
    minv = HUGE
    for v in L:
      if (v < minv):
        minv = v
    \end{verbatim}
- \text{Select}(L, \text{len}(L)/2) \text{ find the median…}
  - How?
    - \text{median} = \text{sorted}(L)[\text{len}(L)/2] \rightarrow O(n \log n)
- \textbf{Can we find medians, or } 1^{\text{st}} \text{ quartiles in } O(n)\text{?}
Select Recursion

- **Select(L, k)** finds the $k^{th}$ smallest element in $L$
  - Select an element $m$ from unsorted list $L$ and partition $L$ the array into two smaller lists:
    - $L_{lo}$ - elements smaller than $m$
    - $L_{hi}$ - elements larger than $m$.

- If $\text{len}(L_{lo}) > k$ then
  Select($L_{lo}$, $k$)

- else if $k > \text{len}(L_{lo}) + 1$ then
  Select($L_{hi}$, $k - \text{len}(L_{lo}) - 1$)

- else $m$ is the $k^{th}$ smallest element
Example of Select(L, 5)

Given an array:  $L = \{ 6, 3, 2, 8, 4, 5, 1, 7, 0, 9 \}$

**Step 1:** Choose the first element as $m$

$L = \{ 6, 3, 2, 8, 4, 5, 1, 7, 0, 9 \}$

Our Selection
Step 2: Split the array into $L_{lo}$ and $L_{hi}$

$L = \{ 6, 3, 2, 8, 4, 5, 1, 7, 0, 9 \}$

$L_{lo} = \{ 3, 2, 4, 5, 1, 0 \}$

$L_{hi} = \{ 8, 7, 9 \}$
Example of Select (cont’d)

**Step 3:** Recursively call Select on either $L_{lo}$ or $L_{hi}$ until $\text{len}(L_{lo}) = k$, then return $m$.

- $\text{len}(L_{lo}) > k = 5 \rightarrow \text{Select}([3, 2, 4, 5, 1, 0], 5)$
  - $m = 3$
  - $L_{lo} = \{2, 1, 0\}$, $L_{hi} = \{4, 5\}$

- $k = 5 > \text{len}(L_{lo}) + 1 \rightarrow \text{Select}([4, 5], 5 - 3 - 1)$
  - $m = 4$
  - $L_{lo} = \{\text{empty}\}$, $L_{hi} = \{5\}$

- $k = 1 = \text{len}(L_{lo}) + 1 \rightarrow \text{return 4}$
def select(L, k):
    value = L[0]
    Llo = [t for t in data if t < value]
    Lhi = [t for t in data if t > value]
    below = len(Llo) + 1
    if (k < len(Llo)):
        return select(Llo, k)
    elif (k > below):
        return select(Lhi, k - below)
    else:
        return value
Select Analysis with Good Splits

- Runtime depends on our selection of $m$:

  - A good selection will split $L$ evenly such that

  $$|L_{lo}| = |L_{hi}| = |L|/2$$

  - The recurrence relation is:

  $$T(n) = T(n/2)$$

  - $n + n/2 + n/4 + n/8 + n/16 + \ldots = 2n \Rightarrow O(n)$

  Same as search for minimum
Select Analysis with Bad Splits

However, a poor selection will split $L$ unevenly and in the worst case, all elements will be greater or less than $m$ so that one Sublist is full and the other is empty.

For a poor selection, the recurrence relation is

$$T(n) = T(n-1)$$

In this case, the runtime is $O(n^2)$.

Our dilemma:

$O(n)$ or $O(n^2)$,

depending on the list… or $O(n \log n)$ independent of it
Select Analysis (cont’d)

- Select seems risky compared to sort
- To improve Select, we need to choose $m$ to give good ‘splits’
- It can be proven that to achieve $O(n)$ running time, we don’t need a perfect splits, just reasonably good ones.
- In fact, if both subarrays are at least of size $n/4$, then running time will be $O(n)$.
- This implies that half of the choices of $m$ make good splitters.
A Randomized Approach

• To improve Select, *randomly* select \( m \).

• Since half of the elements will be good splitters, if we choose \( m \) at random we will get a 50\% chance that \( m \) will be a good choice.

• This approach will make sure that no matter what input is received, the expected running time is small.
def randomizedSelect(L, k):
    value = random.choice(L)
    Llo = [t for t in data if t < value]
    Lhi = [t for t in data if t > value]
    below = len(Llo) + 1
    if (k < len(Llo)):
        return randomizedSelect(Llo, k)
    elif (k > below):
        return randomizedSelect(Lhi, k-below)
    else:
        return value
RandomizedSelect Analysis

• Worst case runtime: $O(n^2)$
• *Expected runtime*: $O(n)$.
• Expected runtime is a good measure of the performance of randomized algorithms, often more informative than worst case runtimes.
• Worst case runtimes are rarely repeated
• RandomizedSelect always returns the correct answer, which offers a way to classify Randomized Algorithms.
Two Types of Randomized Algorithms

• Las Vegas Algorithms – always produce the correct solution (i.e. randomizedSelect)

• Monte Carlo Algorithms – do not always return the correct solution.

• Las Vegas Algorithms are always preferred, but they are often hard to come by.
The Motif Finding Problem

Motif Finding Problem: Given a list of $t$ sequences each of length $n$, find the “best” pattern of length $l$ that appears in each of the $t$ sequences.

$l = 8$

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$n = 69$
A New Motif Finding Approach

• **Motif Finding Problem:** Given a list of \( t \) sequences each of length \( n \), find the “best” pattern of length \( l \) that appears in each of the \( t \) sequences.

• **Previously:** we solved the Motif Finding Problem using a Branch and Bound or a Greedy technique.

• **Now:** randomly select possible locations and find a way to greedily change those locations until we converge to the hidden motif.
Profiles Revisited

• Let $s = (s_1, ..., s_t)$ be the starting positions for $l$-mers in our $t$ sequences.

• The substrings corresponding to these starting positions will form:

  - $t \times l$ alignment matrix
  - $4 \times l$ profile matrix*

* Note that we now define the profile matrix in terms of frequency, not counts as in Lecture 5.

$$P(X|\text{profile}) = 0.6 \times 0.8 \times 0.8 \times 1.0 \times 0.6 \times 0.8 \times 0.6 \times 0.8 = 0.0885$$
Scoring Strings with a Profile

- Let l-mer \( a = a_1, a_2, a_3, \ldots a_l \)
- \( P(a \mid P) \) is defined as the probability that an \( l \)-mer \( a \) was created by the Profile \( P \).
- If \( a \) is very similar to the consensus string of \( P \) then \( P(a \mid P) \) will be high.
- If \( a \) is very different, then \( P(a \mid P) \) will be low.

\[
Prob(a \mid P) = \prod_{i=1}^{l} p(a_i, i)
\]
**Scoring Strings with a Profile (cont’d)**

Given a profile: \( P = \)

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>C</th>
<th>T</th>
<th>G</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1/2</td>
<td>7/8</td>
<td>3/8</td>
<td>0</td>
</tr>
<tr>
<td>C</td>
<td>1/8</td>
<td>0</td>
<td>1/2</td>
<td>5/8</td>
</tr>
<tr>
<td>T</td>
<td>1/8</td>
<td>1/8</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>G</td>
<td>1/4</td>
<td>0</td>
<td>1/8</td>
<td>3/8</td>
</tr>
</tbody>
</table>

The probability of the consensus string:

\[ \text{Prob}(\text{aaacct}|P) = ??? \]
Scoring Strings with a Profile (cont’d)

Given a profile: \( P = \)

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>1/2</th>
<th>7/8</th>
<th>3/8</th>
<th>0</th>
<th>1/8</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1/8</td>
<td>0</td>
<td>1/2</td>
<td>5/8</td>
<td>3/8</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>1/8</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1/4</td>
<td>7/8</td>
<td></td>
</tr>
<tr>
<td>T</td>
<td>1/4</td>
<td>1/8</td>
<td>1/8</td>
<td>3/8</td>
<td>1/4</td>
<td>1/8</td>
<td></td>
</tr>
</tbody>
</table>

The probability of the consensus string:

\[
Prob(\text{aaacct}|P) = 1/2 \times 7/8 \times 3/8 \times 5/8 \times 3/8 \times 7/8 = .033646
\]
Scoring Strings with a Profile (cont’d)

Given a profile: P =

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>7/8</th>
<th>3/8</th>
<th>0</th>
<th>1/8</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1/2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>1/8</td>
<td>0</td>
<td>1/2</td>
<td>5/8</td>
<td>3/8</td>
<td>0</td>
</tr>
<tr>
<td>T</td>
<td>1/8</td>
<td>1/8</td>
<td>0</td>
<td>0</td>
<td>1/4</td>
<td>7/8</td>
</tr>
<tr>
<td>G</td>
<td>1/4</td>
<td>0</td>
<td>1/8</td>
<td>3/8</td>
<td>1/4</td>
<td>1/8</td>
</tr>
</tbody>
</table>

The probability of the consensus string:

\[ \text{Prob}(\text{aaacct} | P) = \frac{1}{2} \times \frac{7}{8} \times \frac{3}{8} \times \frac{5}{8} \times \frac{3}{8} \times \frac{7}{8} = 0.033646 \]

Probability of a different string:

\[ \text{Prob}(\text{atacag} | P) = \frac{1}{2} \times \frac{1}{8} \times \frac{3}{8} \times \frac{5}{8} \times \frac{1}{8} \times \frac{1}{8} = 0.001602 \]
P-Most Probable l-mer

- Define the \( P \)-most probable \( l \)-mer from a sequence as an \( l \)-mer in that sequence which has the highest probability of being created from the profile \( P \).

\[
P = \begin{array}{cccccccc}
A & 1/2 & 7/8 & 3/8 & 0 & 1/8 & 0 \\
C & 1/8 & 0 & 1/2 & 5/8 & 3/8 & 0 \\
T & 1/8 & 1/8 & 0 & 0 & 1/4 & 7/8 \\
G & 1/4 & 0 & 1/8 & 3/8 & 1/4 & 1/8 \\
\end{array}
\]

Given a sequence = ctataaacctttacatc, find the P-most probable \( l \)-mer
P-Most Probable l-mer (cont’d)

<table>
<thead>
<tr>
<th></th>
<th>1/2</th>
<th>7/8</th>
<th>3/8</th>
<th>0</th>
<th>1/8</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1/8</td>
<td>0</td>
<td>1/2</td>
<td>5/8</td>
<td>3/8</td>
<td>0</td>
</tr>
<tr>
<td>C</td>
<td>1/8</td>
<td>1/8</td>
<td>0</td>
<td>0</td>
<td>1/4</td>
<td>7/8</td>
</tr>
<tr>
<td>T</td>
<td>1/4</td>
<td>0</td>
<td>1/8</td>
<td>3/8</td>
<td>1/4</td>
<td>1/8</td>
</tr>
<tr>
<td>G</td>
<td>1/4</td>
<td>0</td>
<td>1/8</td>
<td>3/8</td>
<td>1/4</td>
<td>1/8</td>
</tr>
</tbody>
</table>

Find the $\text{Prob}(a|P)$ of every possible 6-mer:

First try: $\text{ctataaacc}t\text{tacatc}$

Second try: $\text{ctataaacc}t\text{tacatc}$

Third try: $\text{ctataaac}c\text{ttacatc}$

-Continue this process to evaluate every possible 6-mer.
### P-Most Probable l-mer (cont’d)

Compute $prob(a|P)$ for every possible 6-mer:

<table>
<thead>
<tr>
<th>String, Highlighted in Red</th>
<th>Calculations</th>
<th>$prob(a \mid P)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>ctataaacctttacat</td>
<td>$1/8 \times 1/8 \times 3/8 \times 0 \times 1/8 \times 0$</td>
<td>0</td>
</tr>
<tr>
<td>ctataaaacctttacat</td>
<td>$1/2 \times 7/8 \times 0 \times 0 \times 1/8 \times 0$</td>
<td>0</td>
</tr>
<tr>
<td>ctataaaccctttacat</td>
<td>$1/2 \times 1/8 \times 3/8 \times 0 \times 1/8 \times 0$</td>
<td>0</td>
</tr>
<tr>
<td>ctataaaccctttacat</td>
<td>$1/8 \times 7/8 \times 3/8 \times 0 \times 3/8 \times 0$</td>
<td>0</td>
</tr>
<tr>
<td>ctataaaccctttacat</td>
<td>$1/2 \times 7/8 \times 3/8 \times 5/8 \times 3/8 \times 7/8$</td>
<td>0.0336</td>
</tr>
<tr>
<td>ctataaaccctttacat</td>
<td>$1/2 \times 7/8 \times 1/2 \times 5/8 \times 1/4 \times 7/8$</td>
<td>0.0299</td>
</tr>
<tr>
<td>ctataaaccctttacat</td>
<td>$1/2 \times 0 \times 1/2 \times 0 \times 1/4 \times 0$</td>
<td>0</td>
</tr>
<tr>
<td>ctataaaccctttacat</td>
<td>$1/8 \times 0 \times 0 \times 0 \times 1/8 \times 0$</td>
<td>0</td>
</tr>
<tr>
<td>ctataaaccctttacat</td>
<td>$1/8 \times 1/8 \times 0 \times 0 \times 3/8 \times 0$</td>
<td>0</td>
</tr>
<tr>
<td>ctataaaccctttacat</td>
<td>$1/8 \times 1/8 \times 3/8 \times 5/8 \times 1/8 \times 7/8$</td>
<td>0.0004</td>
</tr>
</tbody>
</table>
### P-Most Probable 6-mer (cont’d)

P-Most Probable 6-mer in the sequence is **aaacct**:

<table>
<thead>
<tr>
<th>String, Highlighted in Red</th>
<th>Calculations</th>
<th>( \text{Prob}(a \mid P) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>ctataaacctttacat</td>
<td>1/8 x 1/8 x 3/8 x 0 x 1/8 x 0</td>
<td>0</td>
</tr>
<tr>
<td>ctataaacctttacat</td>
<td>1/2 x 7/8 x 0 x 0 x 1/8 x 0</td>
<td>0</td>
</tr>
<tr>
<td>ctataaacctttacat</td>
<td>1/2 x 1/8 x 3/8 x 0 x 1/8 x 0</td>
<td>0</td>
</tr>
<tr>
<td>ctataaacctttacat</td>
<td>1/8 x 7/8 x 3/8 x 0 x 3/8 x 0</td>
<td>0</td>
</tr>
<tr>
<td><strong>ctataaacctttacat</strong></td>
<td>( 1/2 \times 7/8 \times 3/8 \times 5/8 \times 3/8 \times 7/8 )</td>
<td><strong>.0336</strong></td>
</tr>
<tr>
<td>ctataaacctttacat</td>
<td>1/2 x 7/8 x 1/2 x 5/8 x 1/4 x 7/8</td>
<td><strong>.0299</strong></td>
</tr>
<tr>
<td>ctataaacctttacat</td>
<td>1/2 x 0 x 1/2 x 0 1/4 x 0</td>
<td>0</td>
</tr>
<tr>
<td>ctataaacctttacat</td>
<td>1/8 x 0 x 0 x 0 x 1/8 x 0</td>
<td>0</td>
</tr>
<tr>
<td>ctataaacctttacat</td>
<td>1/8 x 1/8 x 0 x 0 x 3/8 x 0</td>
<td>0</td>
</tr>
<tr>
<td>ctataaacctttacat</td>
<td>1/8 x 1/8 x 3/8 x 5/8 x 1/8 x 7/8</td>
<td><strong>.0004</strong></td>
</tr>
</tbody>
</table>

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P-Most Probable l-mer (cont’d)

**aaacct** is the P-most probable 6-mer in:

```
ctataaccttacatc
```

because $\text{Prob}(\text{aaacct}|P) = .0336$ is greater than the $\text{Prob}(a|P)$ of any other 6-mer in the sequence.
Dealing with Zeroes

• In our toy example \( \text{prob}(a \mid P) = 0 \) in many cases. In practice, there will be enough sequences so that the number of elements in the profile with a frequency of zero is small.

• To avoid many entries with \( \text{prob}(a \mid P) = 0 \), there exist techniques to equate zero to a very small number so that one zero does not make the entire probability of a string zero (assigning a prior probability, we will not address these techniques here).
### P-Most Probable $l$-mers in Many Sequences

- Find the $\mathbf{P}$-most probable $l$-mer in each of the “t” sequences.

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>C</th>
<th>T</th>
<th>G</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>1/2</td>
<td>1/8</td>
<td>1/8</td>
<td>1/4</td>
</tr>
<tr>
<td>0</td>
<td>3/8</td>
<td>5/8</td>
<td>0</td>
<td>1/8</td>
</tr>
<tr>
<td>0</td>
<td>1/8</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1/4</td>
<td>7/8</td>
<td>1/8</td>
</tr>
</tbody>
</table>

$$\mathbf{P} =$$

- ctataaacgttacatc
- atagcgattcgactg
- cagcccagaacccct
- cggtatacctttacatc
- tgcattcaatagctta
- tatcctttccactcac
- ctccaaatccttttaca
- ggtcatccttttatcct
P-Most Probable $l$-mers in Many Sequences (cont’d)

\[
\begin{array}{ccccccc}
1 & a & a & a & c & g & t \\
2 & a & t & a & g & c & g \\
3 & a & a & c & c & c & t \\
4 & g & a & a & c & c & t \\
5 & a & t & a & g & c & t \\
6 & g & a & c & c & t & g \\
7 & a & t & c & c & t & t \\
8 & t & a & c & c & t & t \\
\end{array}
\]

\[
\begin{array}{ccccccc}
A & 5/8 & 5/8 & 4/8 & 0 & 0 & 0 \\
C & 0 & 0 & 4/8 & 6/8 & 4/8 & 0 \\
T & 1/8 & 3/8 & 0 & 0 & 3/8 & 6/8 \\
G & 2/8 & 0 & 0 & 2/8 & 1/8 & 2/8 \\
\end{array}
\]

P-Most Probable $l$-mers form a new profile

catataaacgttacatc
atatagcgattcgactg
cagcccaagaacct
cggtgaacctttacatc
tgcattcaatatgctaatt
tgtcctgtccactcactc
tctccaaatccttttaca
ggtctacctttatatcct
Comparing New and Old Profiles

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>a</th>
<th>a</th>
<th>c</th>
<th>g</th>
<th>t</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>a</td>
<td>t</td>
<td>a</td>
<td>g</td>
<td>c</td>
<td>g</td>
</tr>
<tr>
<td>3</td>
<td>a</td>
<td>a</td>
<td>c</td>
<td>c</td>
<td>c</td>
<td>t</td>
</tr>
<tr>
<td>4</td>
<td>g</td>
<td>a</td>
<td>a</td>
<td>c</td>
<td>c</td>
<td>t</td>
</tr>
<tr>
<td>5</td>
<td>a</td>
<td>t</td>
<td>a</td>
<td>g</td>
<td>c</td>
<td>t</td>
</tr>
<tr>
<td>6</td>
<td>g</td>
<td>a</td>
<td>c</td>
<td>c</td>
<td>t</td>
<td>g</td>
</tr>
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<td>7</td>
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<td>c</td>
<td>c</td>
<td>t</td>
<td>t</td>
</tr>
<tr>
<td>8</td>
<td>t</td>
<td>a</td>
<td>c</td>
<td>c</td>
<td>t</td>
<td>t</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>5/8</th>
<th>5/8</th>
<th>4/8</th>
<th>0</th>
<th>0</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>C</td>
<td>0</td>
<td>0</td>
<td>4/8</td>
<td>6/8</td>
<td>4/8</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>T</td>
<td>1/8</td>
<td>3/8</td>
<td>0</td>
<td>0</td>
<td>3/8</td>
<td>6/8</td>
</tr>
<tr>
<td></td>
<td>G</td>
<td>2/8</td>
<td>0</td>
<td>0</td>
<td>2/8</td>
<td>1/8</td>
<td>2/8</td>
</tr>
</tbody>
</table>

**Red** – frequency increased, **Blue** – frequency decreased
Greedy Profile Motif Search

Use P-Most probable $l$-mers to adjust start positions until we reach a “best” profile; this is the motif.

1) Select random starting positions.
3) Create a profile $P$ from the substrings at these starting positions.
4) Find the $P$-most probable $l$-mer $a$ in each sequence and change the starting position to the starting position of $a$.
5) Compute a new profile based on the new starting positions after each iteration and proceed until we cannot increase the score anymore.
GreedyProfileMotifSearch Algorithm

1. \textbf{GreedyProfileMotifSearch}(DNA, t, n, l)
2. Randomly select starting positions \(s = (s_1, \ldots, s_t)\) from DNA
3. \(bestScore \leftarrow 0\)
4. \textbf{while} \(\text{Score}(s, DNA) > bestScore\)
5. \quad form profile \(P\) from \(s\)
6. \quad \(bestScore \leftarrow \text{Score}(s, DNA)\)
7. \quad \textbf{for} \(i \leftarrow 1 \textbf{ to } t\)
8. \quad \quad Find a \(P\)-most probable \(l\)-mer \(a\) from the \(i^{\text{th}}\) sequence
9. \quad \quad \(s_i \leftarrow \text{starting position of } a\)
10. \textbf{return} \(bestScore\)
GreedyProfileMotifSearch Analysis

• Since we choose starting positions randomly, there is little chance that our guess will be close to an optimal motif, meaning it will take a very long time to find the optimal motif.

• It is unlikely that the random starting positions will lead us to the correct solution at all.

• In practice, this algorithm is run many times with the hope that random starting positions will be close to the optimum solution simply by chance.
GreedyProfileMotifSearch is probably not the best way to find motifs.

However, we can improve the algorithm by introducing Gibbs Sampling, an iterative procedure that discards one $l$-mer after each iteration and replaces it with a new one.

Gibbs Sampling proceeds more slowly and chooses new $l$-mers at random increasing the odds that it will converge to the correct solution.
How Gibbs Sampling Works

1) Randomly choose starting positions \( s = (s_1, \ldots, s_t) \) and form the set of \( l \)-mers associated with these starting positions.

2) Randomly choose one of the \( t \) sequences.

3) Create a profile \( \mathbf{P} \) from the other \( t - 1 \) sequences.

4) For each position in the removed sequence, calculate the probability that the \( l \)-mer starting at that position was generated by \( \mathbf{P} \).

5) Choose a new starting position for the removed sequence at random based on the probabilities calculated in step 4.

6) Repeat steps 2-5 until there is no improvement.
Gibbs Sampling: an Example

Input:

\[ t = 5 \text{ sequences, motif length } l = 8 \]

1. GTAAACAATATTTATAGC
2. AAAATTTACCTCGCAAGG
3. CCGTACTGTCAAGCGTGG
4. TGAGTAAACGACGTCCCA
5. TACTTAACACCCTGTCAA
Gibbs Sampling: an Example

1) Randomly choose starting positions, \( s = (s_1, s_2, s_3, s_4, s_5) \) in the 5 sequences:

\[
\begin{align*}
    s_1 &= 7 & \text{GTAAAC} & \text{AATATTTTA} & \text{TAGC} \\
    s_2 &= 11 & \text{AAAATTTACC} & \text{TTAGAAGG} \\
    s_3 &= 9 & \text{CCGTACTG} & \text{TCAAGCGT} & \text{GG} \\
    s_4 &= 4 & \text{TGA} & \text{GTAAACGA} & \text{CGTCCCA} \\
    s_5 &= 1 & \text{TACTTAAC} & \text{ACCCTG} & \text{TCAA}
\end{align*}
\]
Gibbs Sampling: an Example

2) Choose one of the sequences at random:

**Sequence 2:** AAAATTTTACCTTTAGAAGG

s₁ = 7  GTAAACAAATAATTATTATAGC
s₂ = 11  AAAATTTTACCTTTAGAAGG
s₃ = 9  CCGTACTGTCGAACGTGG
s₄ = 4  TGAGTAAACGACGTCCCA
s₅ = 1  TACTTTAAACACCCTGTCAAA
Gibbs Sampling: an Example

2) Choose one of the sequences at random:

Sequence 2: AAAATTTTACCTTTAGAAGG

\[ s_1 = 7 \quad \text{GTAAAC} \textcolor{red}{\text{AATATT}} \textcolor{red}{\text{TATAGC}} \]

\[ s_3 = 9 \quad \text{CCGTACTG} \textcolor{red}{\text{TCAAGCGT}} \textcolor{red}{\text{GG}} \]

\[ s_4 = 4 \quad \text{TGA} \textcolor{red}{\text{GTAAACGACGTC}} \textcolor{red}{\text{CCA}} \]

\[ s_5 = 1 \quad \text{TACTTTAAC} \textcolor{red}{\text{ACCCTGTC}} \textcolor{red}{\text{AA}} \]

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Gibbs Sampling: an Example

3) Create profile $P$ from $l$-mers in remaining 4 sequences:

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>A</th>
<th>T</th>
<th>A</th>
<th>T</th>
<th>T</th>
<th>T</th>
<th>A</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>A</td>
<td>A</td>
<td>T</td>
<td>A</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>A</td>
</tr>
<tr>
<td>3</td>
<td>T</td>
<td>C</td>
<td>A</td>
<td>A</td>
<td>G</td>
<td>C</td>
<td>G</td>
<td>T</td>
</tr>
<tr>
<td>4</td>
<td>G</td>
<td>T</td>
<td>A</td>
<td>A</td>
<td>A</td>
<td>C</td>
<td>G</td>
<td>A</td>
</tr>
<tr>
<td>5</td>
<td>T</td>
<td>A</td>
<td>C</td>
<td>T</td>
<td>T</td>
<td>A</td>
<td>A</td>
<td>C</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>C</th>
<th>T</th>
<th>G</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1/4</td>
<td>2/4</td>
<td>2/4</td>
<td>3/4</td>
</tr>
<tr>
<td>C</td>
<td>0</td>
<td>1/4</td>
<td>1/4</td>
<td>0</td>
</tr>
<tr>
<td>T</td>
<td>2/4</td>
<td>1/4</td>
<td>1/4</td>
<td>1/4</td>
</tr>
<tr>
<td>G</td>
<td>1/4</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Consensus String: T A A A T C G A
4) Calculate the $\text{prob}(a \mid P)$ for every possible 8-mer in the removed sequence:

Strings Highlighted in Red

<table>
<thead>
<tr>
<th>String</th>
<th>$\text{prob}(a \mid P)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAAATTTACCTTGAAGG</td>
<td>.000732</td>
</tr>
<tr>
<td>AAAATTTACCTTGAAGG</td>
<td>.000122</td>
</tr>
<tr>
<td>AAAATTTACCCTTGAAGG</td>
<td>0</td>
</tr>
<tr>
<td>AAAATTTACCCTTGAAGG</td>
<td>0</td>
</tr>
<tr>
<td>AAAATTTACCTTGAAGG</td>
<td>0</td>
</tr>
<tr>
<td>AAAATTTACCCTTGAAGG</td>
<td>0</td>
</tr>
<tr>
<td>AAAATTTACCTTGAAGG</td>
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</tr>
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<td>AAAATTTACCTTGAAGG</td>
<td>0</td>
</tr>
<tr>
<td>AAAATTTACCTTGAAGG</td>
<td>0.000183</td>
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<tr>
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</tr>
<tr>
<td>AAAATTTACCTTGAAGG</td>
<td>0</td>
</tr>
<tr>
<td>AAAATTTACCTTGAAGG</td>
<td>0</td>
</tr>
</tbody>
</table>
Gibbs Sampling: an Example

5) Create a distribution of probabilities of \( l \)-mers \( \text{prob}(a \mid P) \), and randomly select a new starting position based on this distribution.

A) To create this distribution, divide each probability \( \text{prob}(a \mid P) \) by the total:

- Starting Position 1: \( \text{prob}(\text{AAAATTTA} \mid P) = .706 \)
- Starting Position 2: \( \text{prob}(\text{AAATTTAC} \mid P) = .118 \)
- Starting Position 8: \( \text{prob}(\text{ACCTTAGA} \mid P) = .176 \)
Gibbs Sampling: an Example

B) Select a new starting position at random according to computed distribution:

\[
\begin{align*}
P(\text{selecting starting position 1}): & \quad .706 \\
P(\text{selecting starting position 2}): & \quad .118 \\
P(\text{selecting starting position 8}): & \quad .176
\end{align*}
\]

\[
t = \text{random.random()} \\
\text{if } (t < .706): \\
\quad \# \text{ use position 1} \\
\text{elif } (t < (.706 + .118)): \\
\quad \# \text{ use position 2} \\
\text{else:} \\
\quad \# \text{ use position 8}
\]
Gibbs Sampling: an Example

Assume we select the substring with the highest probability – then we are left with the following new substrings and starting positions.

\[
\begin{align*}
    s_1 &= 7 & \text{GTAAAC} & \text{AAT} & \text{TTT} & \text{TATA} & \text{TAGC} \\
    s_2 &= 1 & \text{AAAATT} & \text{TACCTCGCA} & \text{AGG} \\
    s_3 &= 9 & \text{CCGTACTG} & \text{TCAAGCGTG} & \text{GTG} \\
    s_4 &= 5 & \text{TGAGTAATCGACGTG} & \text{CCC} \\
    s_5 &= 1 & \text{TACTTCACACCCTGTGTCA} & \text{A}
\end{align*}
\]
Gibbs Sampling: an Example

6) We iterate the procedure again with the above starting positions until we cannot improve the score any more.
Gibbs Sampler in Practice

- Gibbs sampling needs to be modified when applied to samples with biased distributions of nucleotides (*relative entropy* approach).
- Gibbs sampling often converges to locally optimal motifs rather than globally optimal motifs.
- Must be run with many randomly chosen seeds to achieve good results.
Another Randomized Approach

- **Random Projection Algorithm** is a different way to solve the Motif Finding Problem.
- **Guiding principle:** Instances of a motif agree at a subset of positions.
- However, it is unclear how to find these “non-mutated” positions.
- To bypass the effect of mutations within a motif, we randomly select a subset of positions in the pattern creating a **projection** of the pattern.
- Search for that projection in a hope that the selected positions are not affected by mutations in most instances of the motif.
Projections

- Choose $k$ positions in string of length $l$.
- Concatenate nucleotides at chosen $k$ positions to form $k$-tuple.
- This can be viewed as a projection of $l$-dimensional space onto $k$-dimensional subspace.

Example:

$l = 15$

Projection

<table>
<thead>
<tr>
<th>ATGGCATTCAGATTC</th>
</tr>
</thead>
</table>

Projection = (2, 4, 5, 7, 11, 12, 13)
Random Projections Algorithm

• Select $k$ out of $l$ positions uniformly at random.

• For each $l$-tuple in input sequences, hash into buckets based on the $k$ selected positions.

• Recover motif from enriched buckets that contain many $l$-tuples with at least one from each sequence.
Random Projections Algorithm (cont’d)

- Some projections will fail to detect motifs but if we try many of them the probability that one of the buckets fills increases.
- In the example below, the bucket **GC*AC is “bad” while the bucket AT**G*C is “good”
Example

- \( l = 7 \) (motif size), \( k = 4 \) (projection size)
- Choose projection \((1, 2, 5, 7)\)

\[ \ldots \text{TAGAC} \text{ATCCGAC} \text{TTGCCTTAC} \text{TAC} \ldots \]

\[ \begin{array}{c}
\text{Buckets} \\
\begin{array}{c}
\text{ATGC} \\
\text{ATCCGAC} \\
\text{ATGCC} \\
\text{GCCTTAC} \\
\text{GCTTC} \\
\end{array}
\end{array} \]
Hashing and Buckets

- Hash function $h(x)$ obtained from $k$ positions of projection.
- Buckets are labeled by values of $h(x)$.
- *Enriched buckets*: contain more than $s$ $l$-tuples, for some parameter $s$ with representatives from all sequences

```
ATGC
GCTC
CATC
ATTC
```
Motif Refinement

• How do we recover the motif from the sequences in enriched buckets?
• $k$ nucleotides are exact matches, (hash key of bucket).
• Use information in other $l-k$ positions as starting point for local refinement scheme, e.g. Gibbs sampler.

ATGC

Local refinement algorithm

ATCCGAC
ATGAGGC
ATAAGTC
ATGCGAC

ATGC

ATGCGAC
Candidate motif
• Random Projection is a procedure for finding good starting points: every enriched bucket is a potential starting point.

• Feeding these starting points into existing algorithms (like Gibbs sampler) provides good local search in vicinity of every starting point.

• These algorithms work particularly well for “good” starting points.
Building Profiles from Buckets

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>C</th>
<th>G</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>ATCCGAC</td>
<td>1.0</td>
<td>0</td>
<td>0.25</td>
<td>0.50</td>
</tr>
<tr>
<td>ATGAGGC</td>
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<td>0</td>
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<td>0.25</td>
</tr>
<tr>
<td>ATAAAGTC</td>
<td>0</td>
<td>0</td>
<td>0.50</td>
<td>0</td>
</tr>
<tr>
<td>ATGTGAC</td>
<td>0</td>
<td>1</td>
<td>0.25</td>
<td>0.25</td>
</tr>
</tbody>
</table>

Profile P

Gibbs sampler

Refined profile P*
Motif Refinement

• For each bucket $h$ containing more than $s$ sequences, form profile $P(h)$

• Use Gibbs sampler algorithm with starting point $P(h)$ to obtain refined profile $P^*$
Random Projection Algorithm

A Single Iteration:

- Choose a random $k$-projection.
- Hash each $l$-mer $x$ in input sequence into bucket labeled by $h(x)$
- From each enriched bucket (e.g., a bucket with more than $s$ sequences), form profile $P$ and perform Gibbs sampler motif refinement
- Candidate motif is best found by selecting the best motif among refinements of all enriched buckets.
Choosing Projection Size

- Projection size $k$
  - choose $k$ small enough so that several motif instances hash to the same bucket.
    \[ k << l, \quad l/2 < k < l - const \]
  - choose $k$ large enough to avoid contamination by spurious $l$-mers:
    \[ 4^k \gg t \left( n - l + 1 \right) \]
It’s Over

• Final Thursday, 4/30
  – 12:00-3:00PM
  – This room: SN011
  – Open book, open notes, no computer, tablet, eReader, or smartphone
  – Will covers material since midterm
  – Study session? Sunday or Monday?