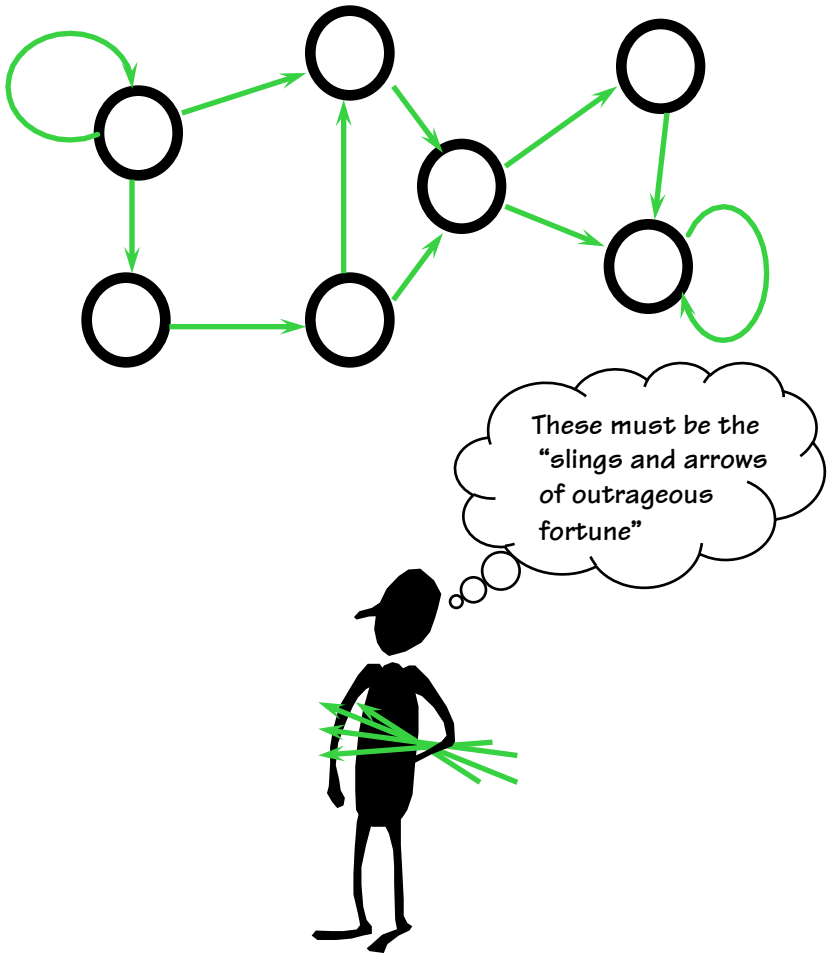
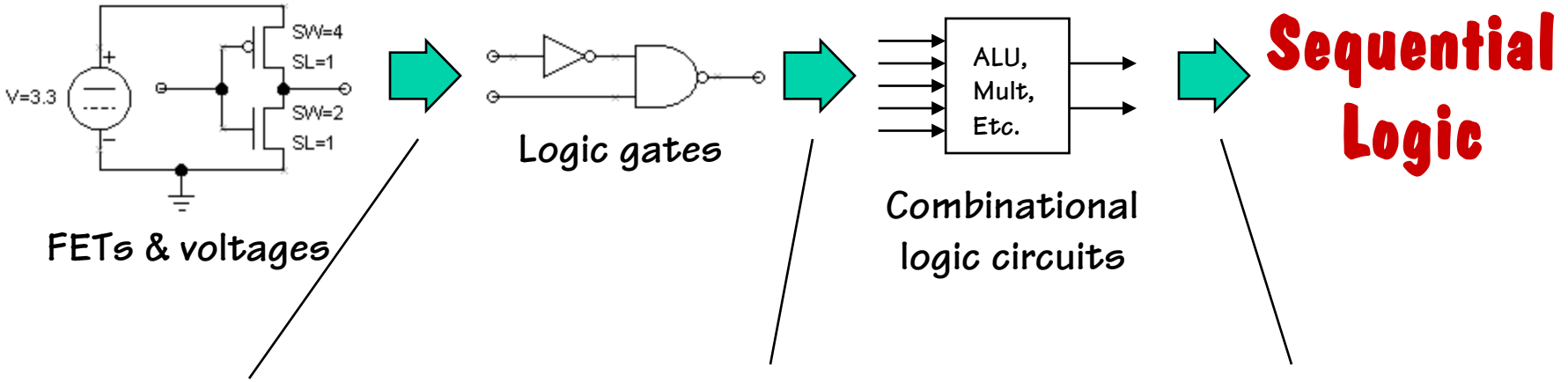
Synchronous Logic

1) Sequential Logic
2) Synchronous Design
3) Synchronous Timing Analysis
4) Single Clock Design
5) Finite State Machines
6) Turing Machines
7) What it means to be "Computable"



Road Traveled So Far…



Combinational contract:

- Voltage-based "bits"
- 1-bit per wire
- Generate quality outputs, tolerate inferior inputs
- Combinational contract
- Complete in/out/timing spec

Acyclic connections
Composable blocks
Design:

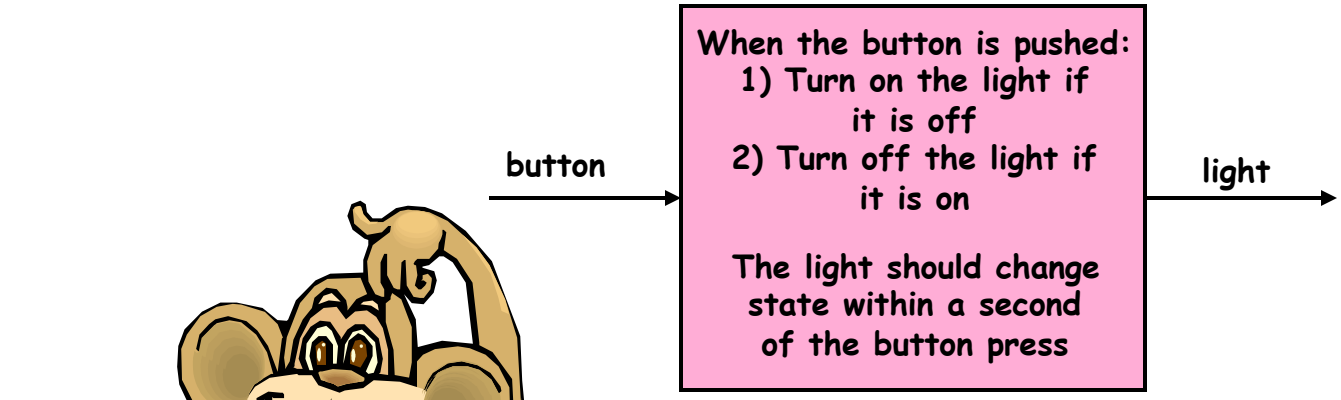
- truth tables
- sum-of-products
- muxes
- ROMs

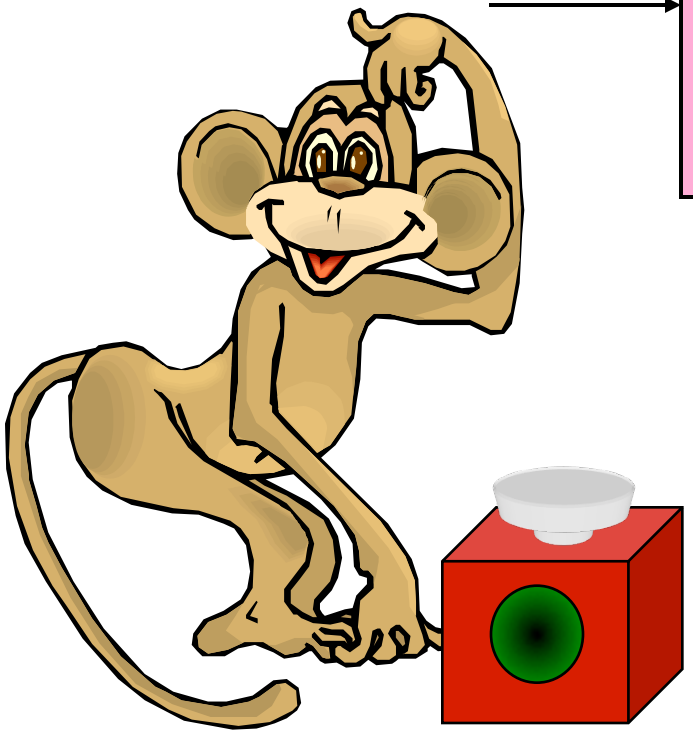
Storage & state
Dynamic discipline
Finite-state machines
Throughput & latency
Pipelining

Our motto: Sweat the details once, and then put a box around it!

Something We Can't Build (Yet)

What if you were given the following system design specification?

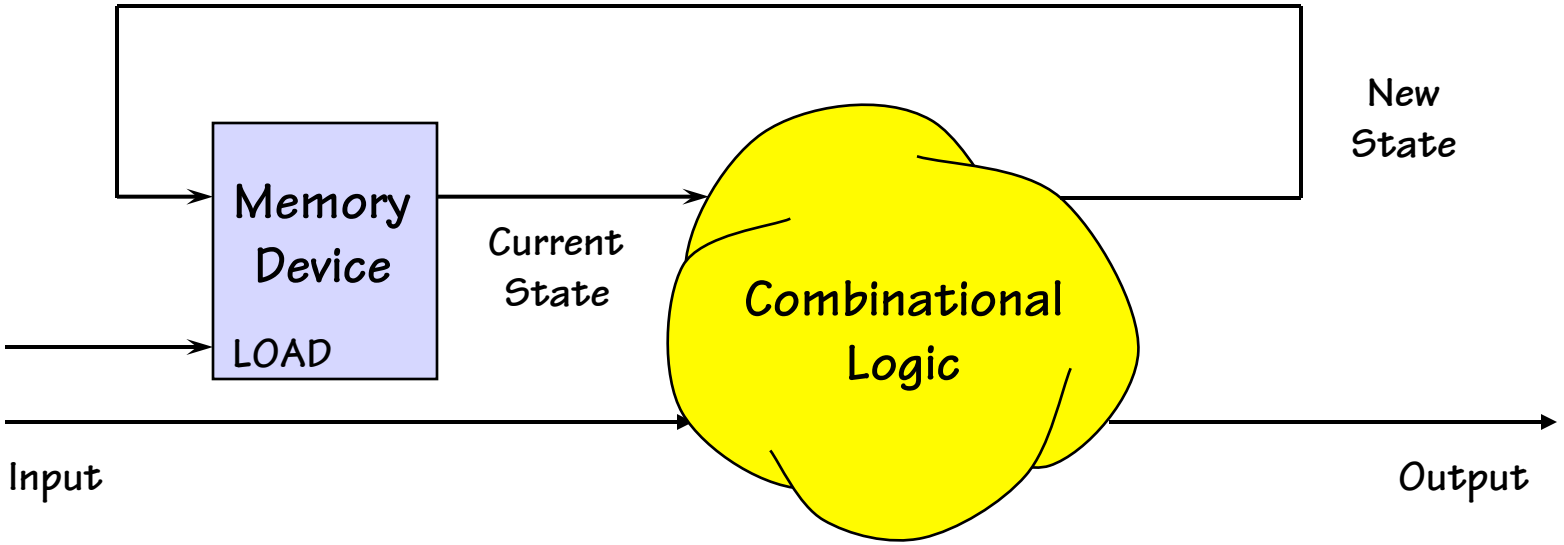




What makes this System so different from those we've discussed before?

1. "State" – i.e. the circuit has memory
2. The output was changed by a input "event" (pushing a button) rather than an input "value"

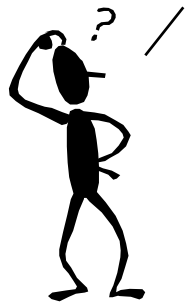
"Sequential" = Stateful



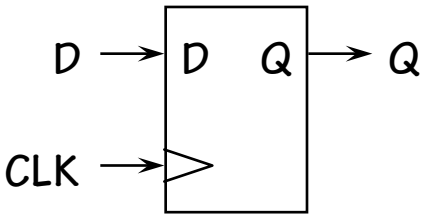
Plan: Build a Sequential Circuit with stored digital STATE –

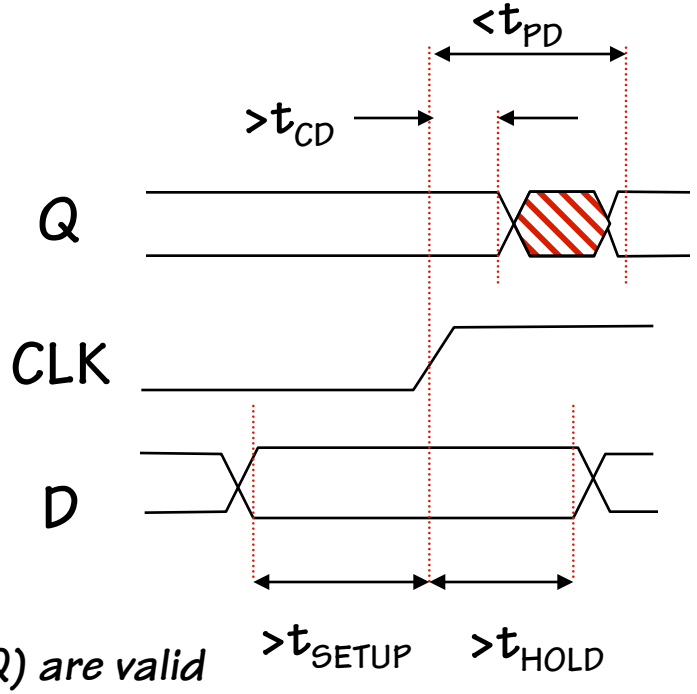
- MEMORY stores CURRENT state
- Combinational Logic computes
  - the NEXT state (Based on inputs & current state)
  - the OUTPUTs (Based on inputs and/or current state)
- State changes on LOAD control input

Didn't we develop some memory devices last time?



Review of Flip Flop Timing





t_{PD}: maximum propagation delay, CLK →Q

How LONG after clock rises until outputs (Q) are valid

t_{CD}: minimum contamination delay, CLK →Q

How SOON after clock rises until outputs (Q) go invalid

We haven't explicitly mentioned this timing attribute, but it must have existed even for combinational logic. We can always safely assume it is 0 (i.e. the outputs become invalid immediately)

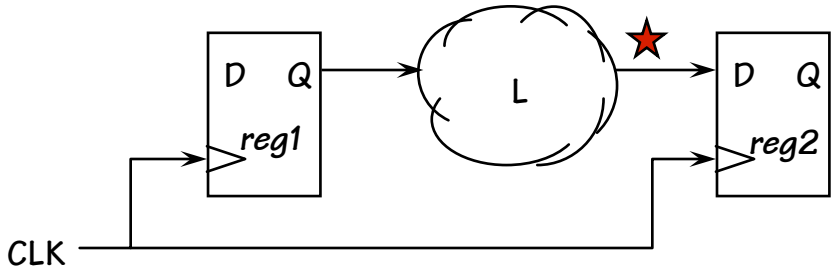
t_{SETUP}: setup time

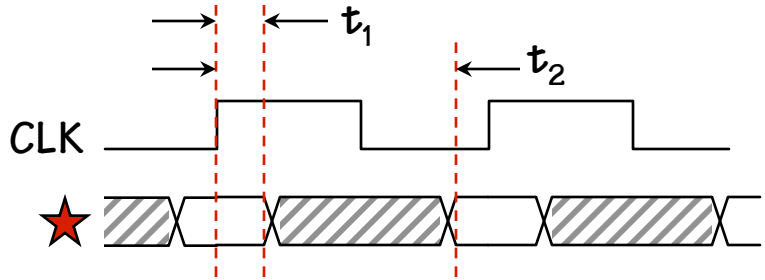
How LONG data (D) input must be stable before clock's rising edge

t_{HOLD}: hold time

How LONG data (D) inputs must be held after clock's rising edge

"Synchronous" Timing Analysis





Questions for register-based designs:

- How much time for useful work (i.e. for combinational logic delay)?
- Does it help to guarantee a minimum t_{CD}? How 'bout designing registers so that

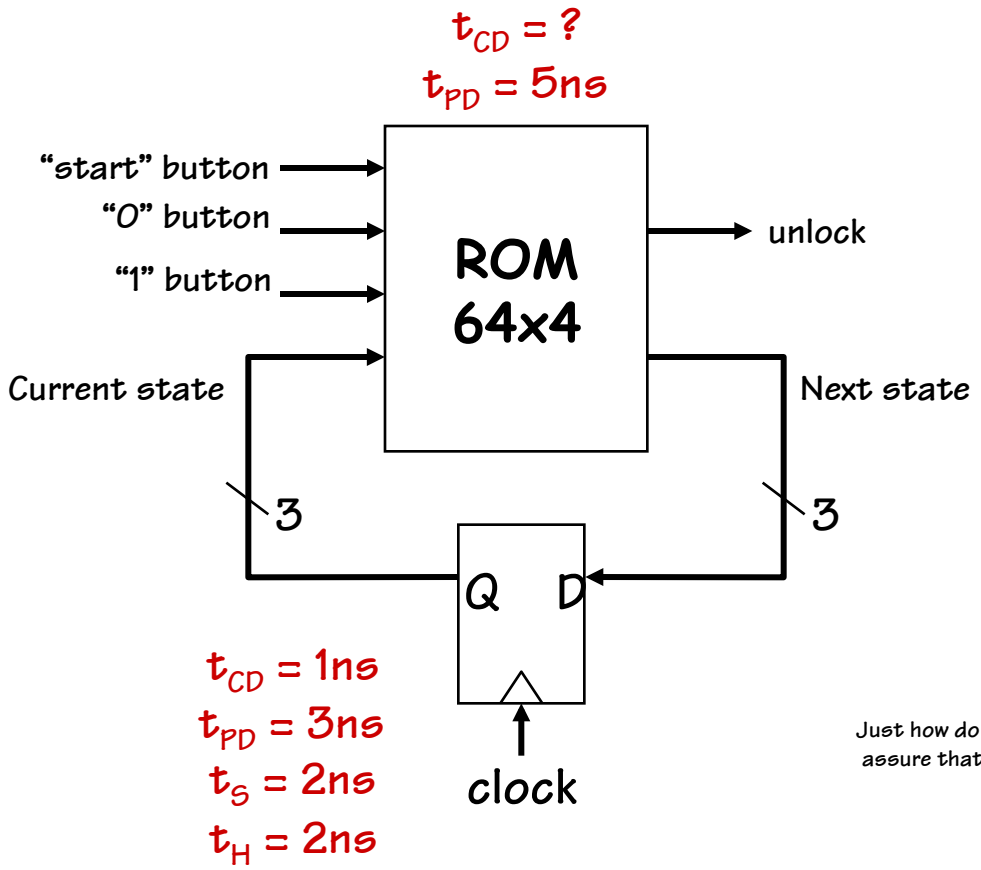
  $t_{CD,reg} > t_{HOLD,reg}$?
- What happens if CLK signal doesn't arrive at the two registers at exactly the same time (a phenomenon known as "clock skew")?

$t_1 = t_{CD,reg1} + t_{CD,L} > t_{HOLD,reg2}$

$t_2 = t_{PD,reg1} + t_{PD,L} < t_{CLK} - t_{SETUP,reg2}$

Minimum Clock Period : $t_{CLK} > t_{PD,reg1} + t_{PD,L} + t_{SETUP,reg2}$

Example: Flip Flop Timing



Questions:

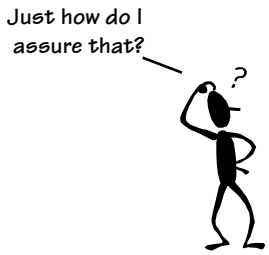
1. t_{CD} for the ROM?

$$t_{CD,REG} + t_{CD,ROM} > t_{H,REG}$$
$$1\text{ ns} + t_{CD,ROM} > 2\text{ nS}$$
$$t_{CD,ROM} > 1\text{ nS}$$

2. Min. clock period?

$$t_{CLK} > t_{PD,REG} + t_{PD,ROM} + t_{S,REG}$$
$$t_{CLK} > 3\text{ ns} + 5\text{ ns} + 2\text{ nS}$$
$$t_{CLK} > 10\text{ nS}$$

3. Constraints on inputs?

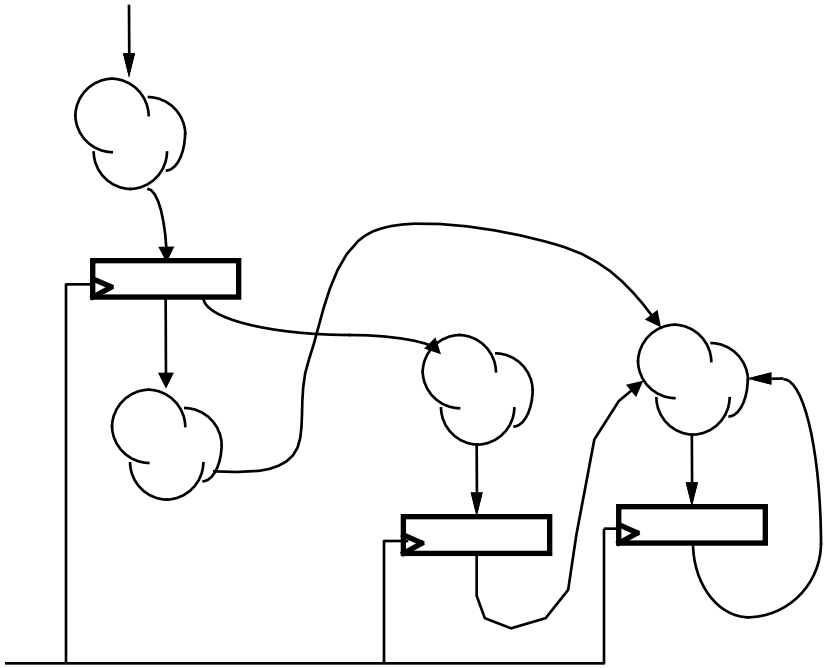


"start" , "0", and "1" must be valid $t_{PD,ROM} + t_{S,REG} = 5 + 2 = 7$ ns before the clock and held $t_{H,REG} - t_{CD,ROM} = 2 - 1 = 1$ ns after it.

Single Synchronous Clock Design

Sequential ≠ Synchronous

However, Synchronous = A recipe for robust sequential circuits:



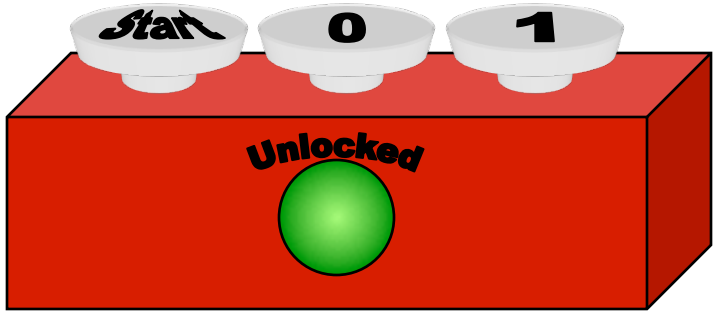
- **No combinational cycles**
  (other than those already built into the registers)
- **Only cares about values of combinational circuits just before rising edge of clock**
- **Clock period greater than every combinational delay**
- **Changes state after all logic transitions have stopped!**

Designing Sequential Logic

Sequential logic is used when the solution to some design problem involves a *sequence* of steps:

How to open digital combination lock w/ 3 buttons ("start", "0" and "1"):

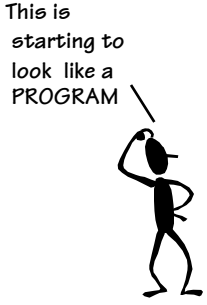
Step 1: press "start" button
Step 2: press "0" button
Step 3: press "1" button
Step 4: press "1" button
Step 5: press "0" button



Information remembered between steps is called state. Might be just what step we're on, or might include results from earlier steps we'll need to complete a later step.

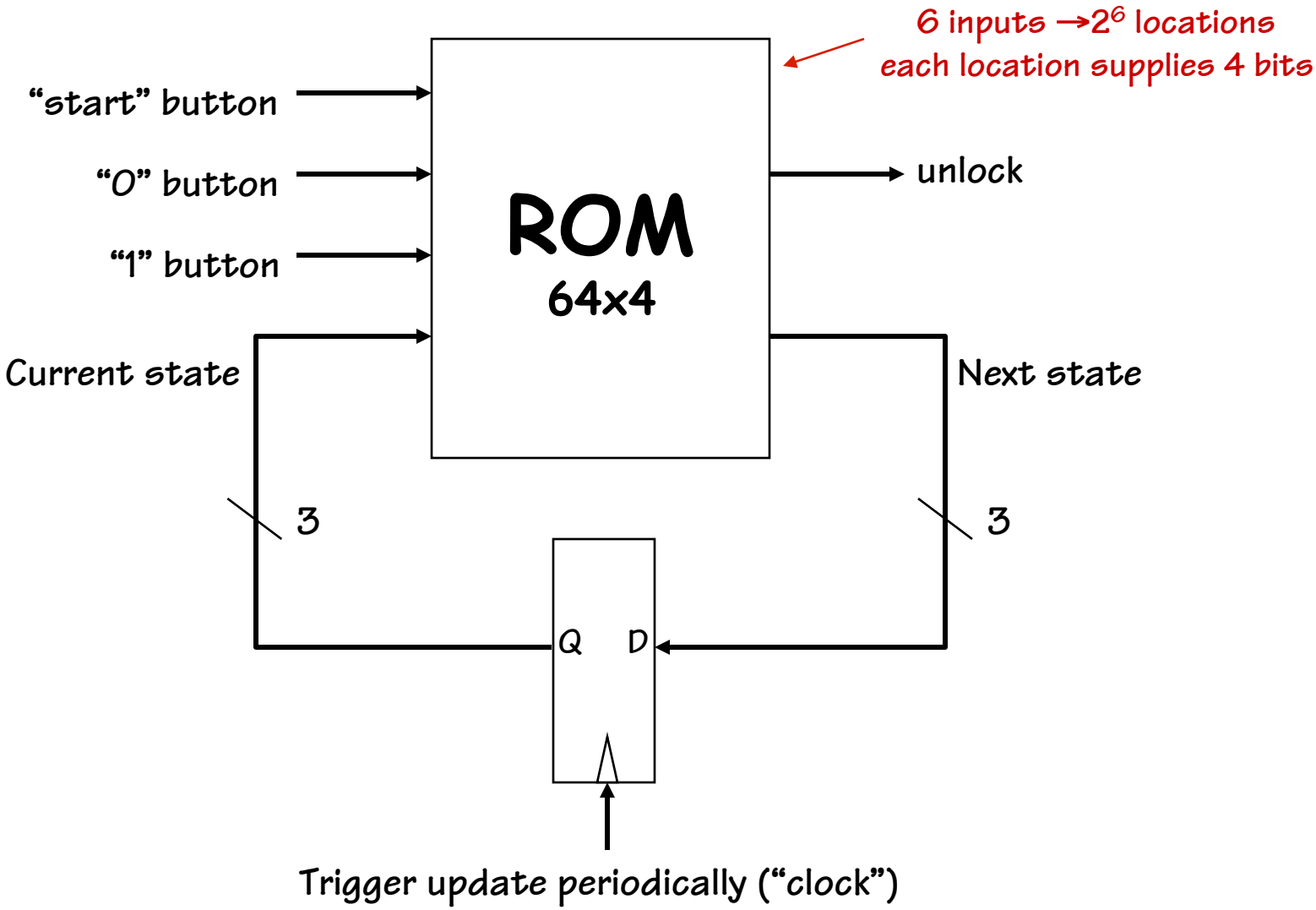
Implementing a "State Machine"

Current state		"start"	"1"	"0"	Next state		unlock
---		1	---	---	start	000	0
start	000	0	0	1	digit1	001	0
start	000	0	1	0	error	101	0
start	000	0	0	0	start	000	0
digit1	001	0	1	0	digit2	010	0
digit1	001	0	0	1	error	101	0
digit1	001	0	0	0	digit1	001	0
digit2	010	0	1	0	digit3	011	0
...							
digit3	011	0	0	1	unlock	100	0
...							
unlock	100	0	1	0	error	101	1
unlock	100	0	0	1	error	101	1
unlock	100	0	0	0	unlock	100	1
error	101	0	---	---	error	101	0

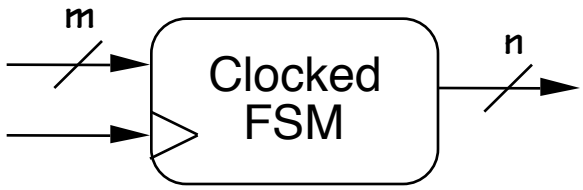


6 different states → encode using 3 bits

Now Do It With Hardware!



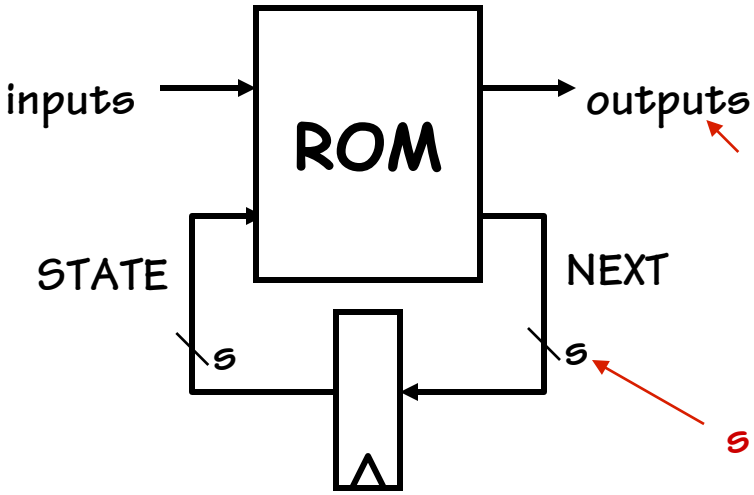
Abstraction du jour: Finite State Machines



A FINITE STATE MACHINE has

- k STATES S_1 ... S_k (one is "initial" state)
- m INPUTS I_1 ... I_m
- n OUTPUTS O_1 ... O_n
- Transition Rules S'(S,i) for each state S and input i
- Output Rules Out(S) for each state S

Discrete State, Time



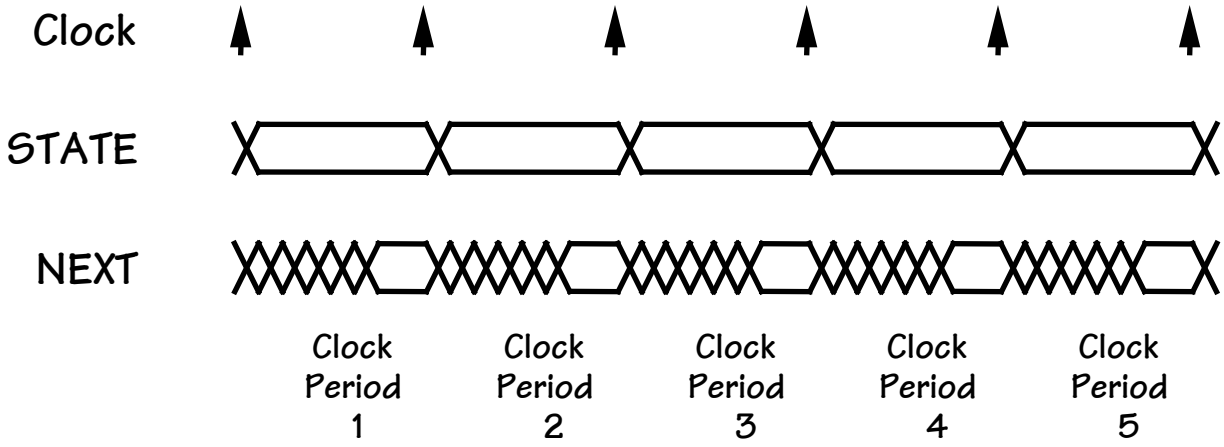
While a ROM is shown here in the feedback path any form of combinational logic can be used to construct a state machine.

Two design choices:

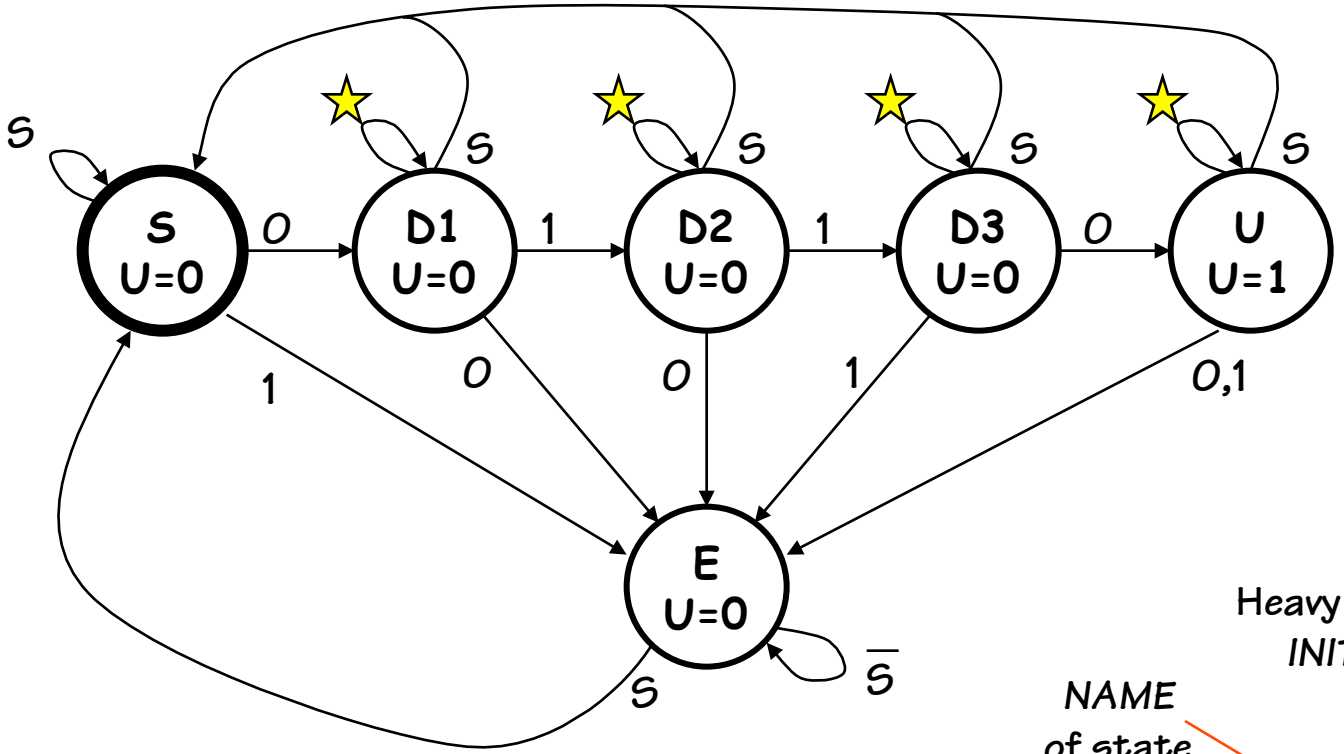
(1) outputs *only* depend on state (Moore)

(2) outputs depend on inputs + state (Mealy)

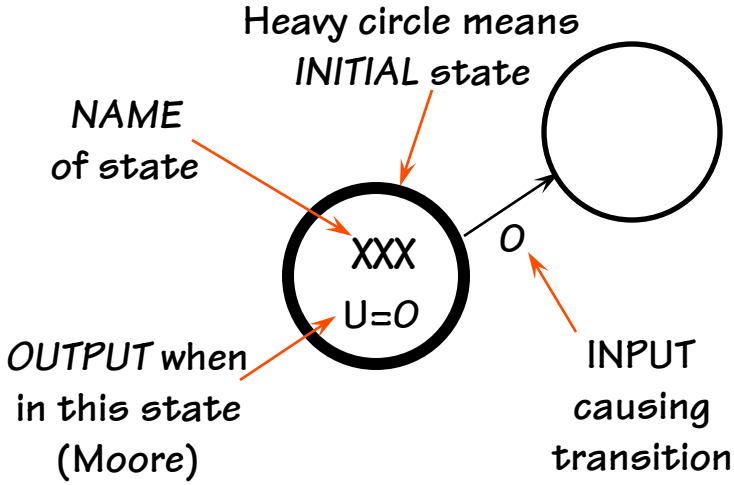
s state bits → 2^s possible states



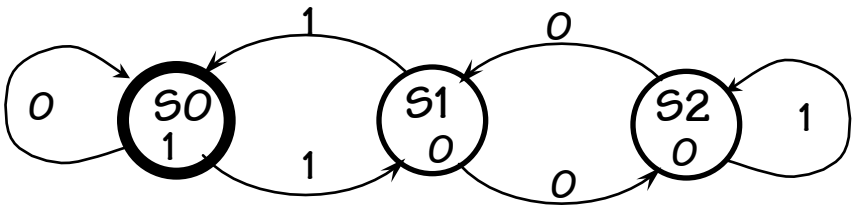
State Transition Diagrams



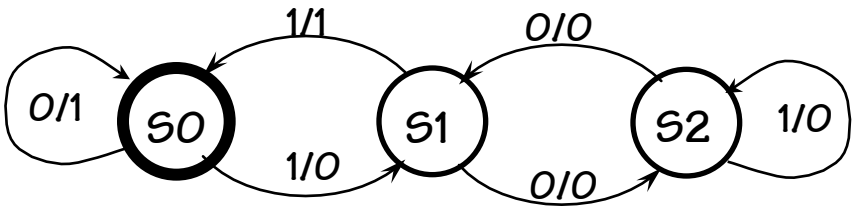
A state transition diagram is an abstract "graph" representation of a state machine, where each state is represented as a node and each transition is represented as a as an arc. It represents the machine's behavior not its implementation.



Valid State Diagrams



MOORE Machine:
Outputs on States



MEALY Machine:
Outputs on Transitions

Arcs leaving a state must be:

(1) mutually exclusive

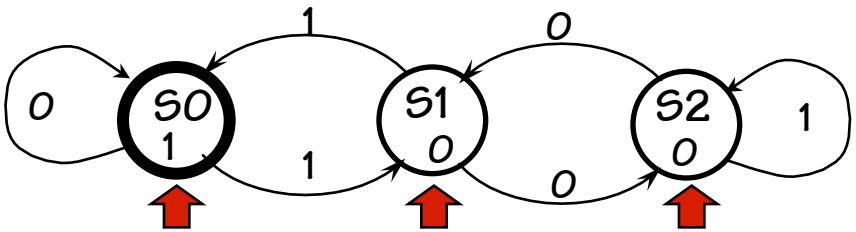
can only have one choice for any given input value

(2) collectively exhaustive

every state must specify what happens for each possible input combination. "Nothing happens" means arc back to itself.

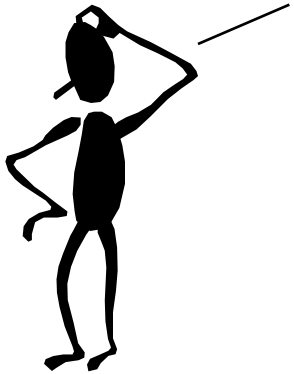
Let's Play State Machine

Let's emulate the behavior specified by the state machine shown below when processing the following string from LSB to MSB.



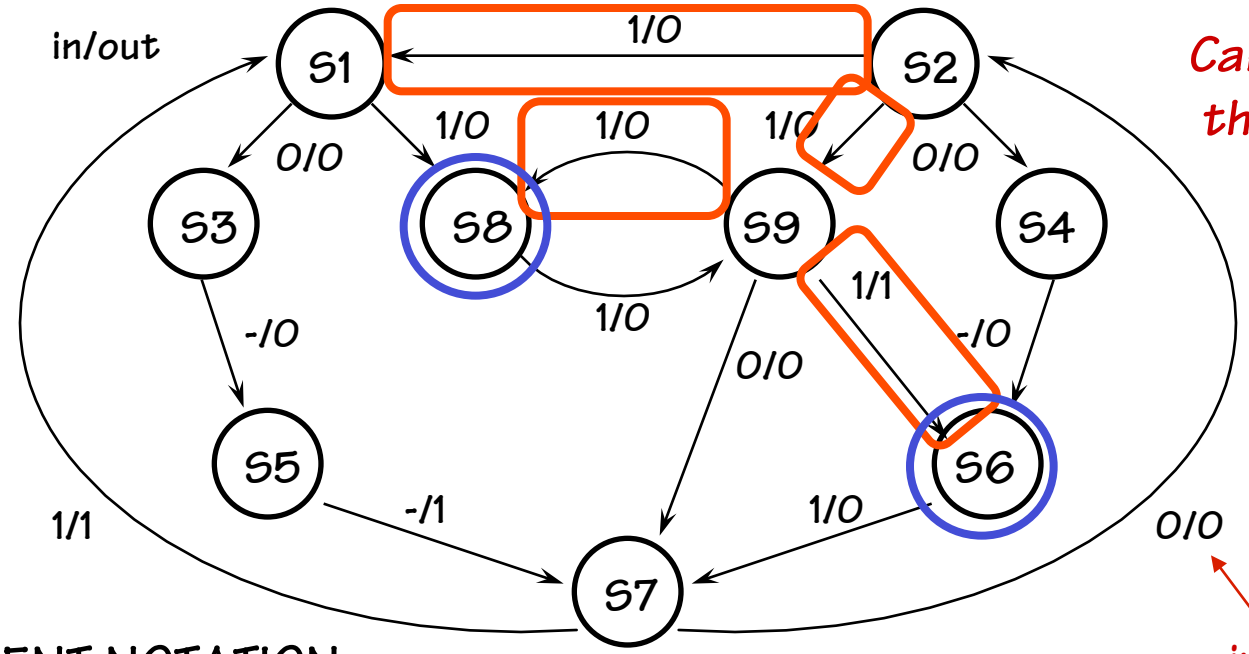
$$39_{10} = 0100111_2$$

	State	Input	Next	Output
T=0	S0	1	S1	0
T=1	S1	1	S0	1
T=2	S0	1	S1	0
T=3	S1	0	S2	0
T=4	S2	0	S1	0
T=5	S1	1	S0	1
T=6	S0	0	S0	1



It looks to me like this machine outputs a 1 whenever the bit sequence that it has seen thus far is a multiple of 3. (this might be useful for my problem set!)

Busted Stuff



Can you spot the problems?

input/output (Mealy)

CONVENIENT NOTATION:

When a transition is made on the next input regardless of its value the arc can be labeled with an X or -

AMBIGOUS TRANSITIONS (Mutual Exclusive property violated):

For each input there can only be one arc leaving a state

UNSPECIFIED TRANSITIONS (Collectively Exhaustive property violated):

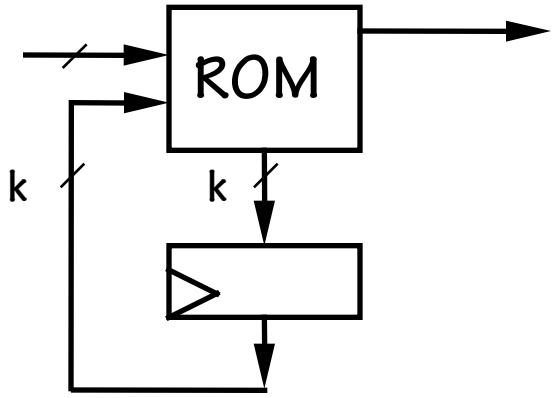
There must be an arc leaving a state for all valid inputs

(It can, however, loop back to the same state)

FSM Party Games

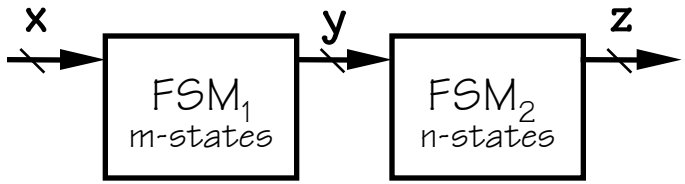
1. What can you say about the number of states?

$$\text{States} \leq 2^k$$

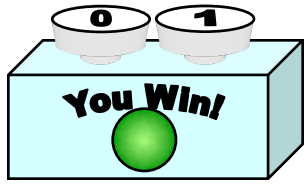


2. Same question:

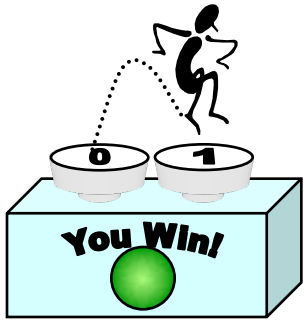
$$\text{States} \leq m \times n$$



3. Here's an FSM. Can you discover its rules?

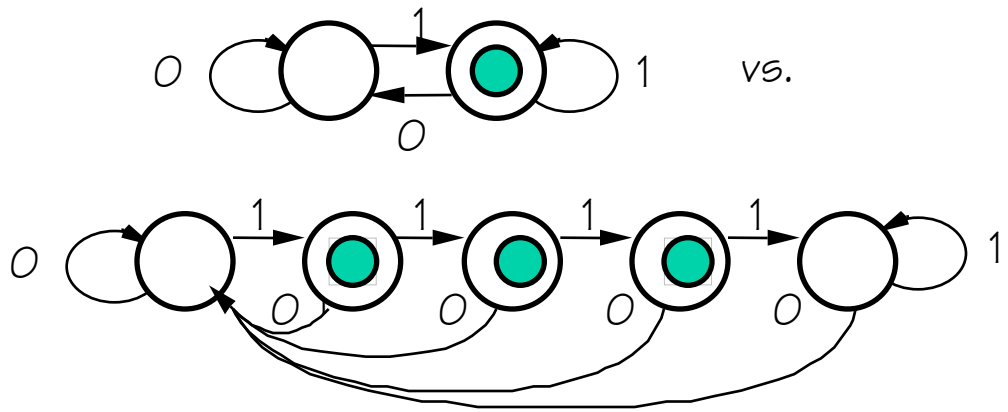


What's My Transition Diagram?



0=OFF,
1=ON?

"1111" =
Surprise!

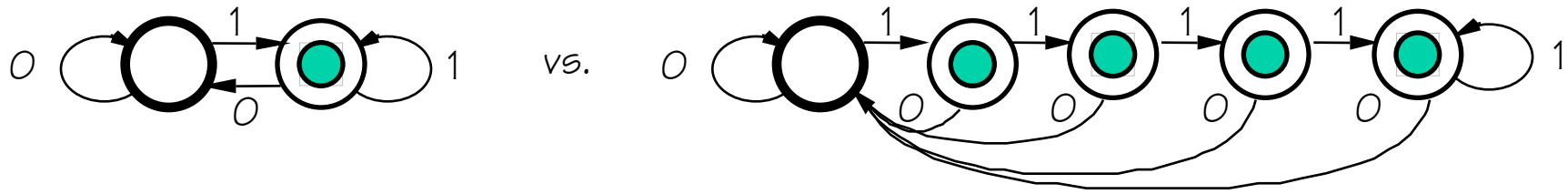


- If you know NOTHING about the FSM, you're never sure!
- If you have a BOUND on the number of states, you can discover its behavior:

  K-state FSM: Every (reachable) state can be reached in < 2^i x k steps.

BUT ... states may be equivalent!

FSM Equivalence



ARE THEY DIFFERENT?

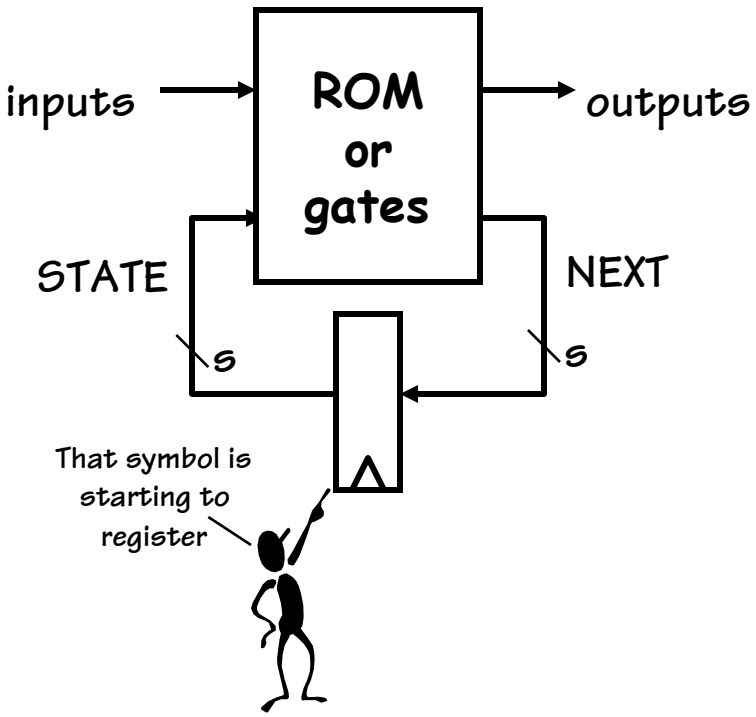
NOT in any practical sense! They are EXTERNALLY INDISTINGUISHABLE, hence interchangeable.

> FSMs are *EQUIVALENT* iff every input sequence yields identical output sequences.

ENGINEERING GOAL:

- HAVE an FSM which *works*...
- WANT <u>simplest</u> (ergo cheapest) equivalent FSM.

Housekeeping issues...



1. Initialization? Clear the memory?

2. Unused state encodings?
   - waste ROM (use PLA or gates)
   - meaning?

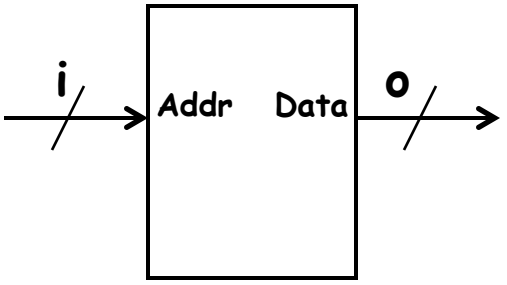
3. Synchronizing input changes with state update?

4. Choosing encoding for state?

2-Flavors of Processing Elements

Combinational Logic:

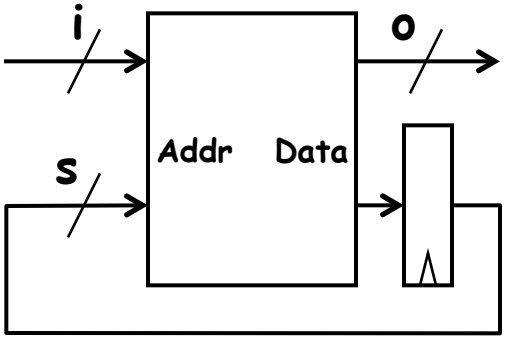
Table look-up, ROM



Recall that there are precisely
2^{2^i}, i-input combinational functions.
A single ROM can store 'o' of them.

Finite State Machines:

ROM with State Memory



Thus far, we know of nothing more powerful than an FSM

FSMs as Programmable Machines

ROM-based FSM sketch:

Given i, s, and o,
we need a ROM organized as:

2^{i+s} words x (o+s) bits

So how many possible
i-input,
o-output,
FSMs with
s-state bits
exist?

The number of "bits" in the ROM

All possible settings of the ROM's contents to: 1 or 0

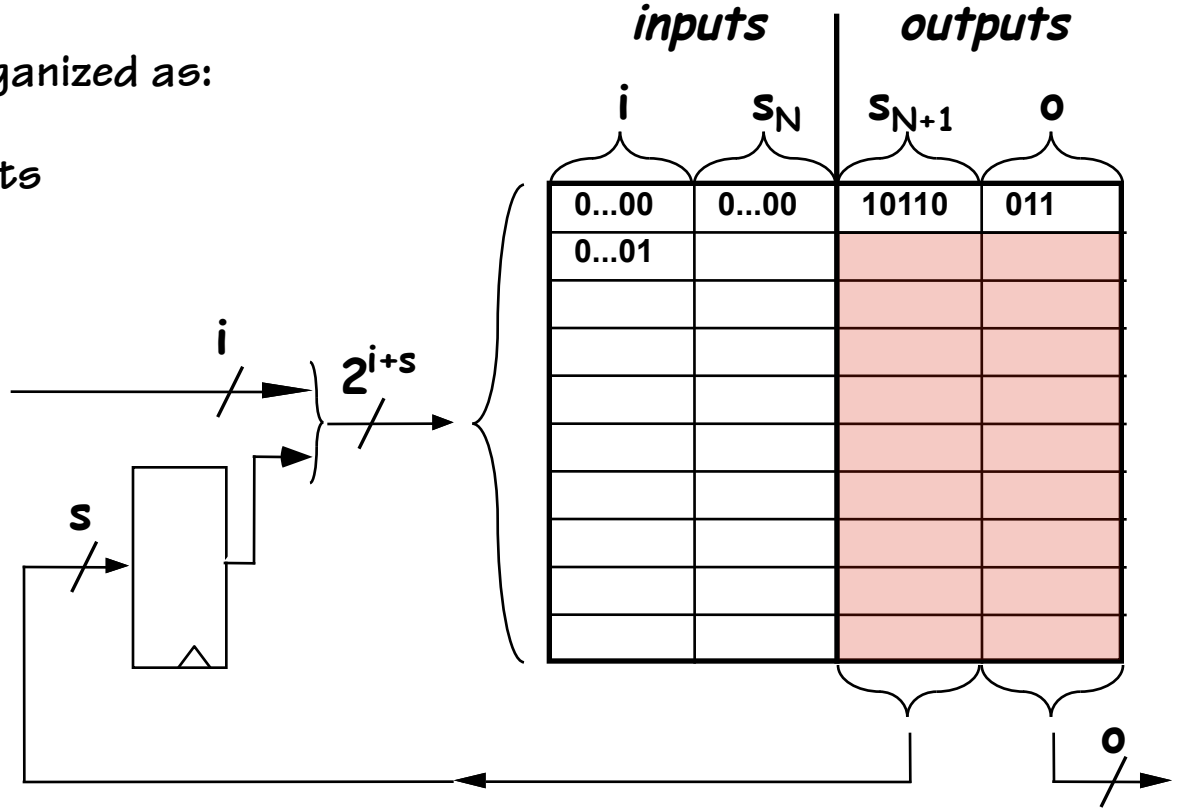
$$2^{(o+s)2^{i+s}}$$

(some may be equivalent)

How many state machines are there with 1-input, 1-output, and 1 state bit?

$2^{(1+1)4}=2^8=256$

An FSM's behavior is completely determined by its ROM contents.



Recall how we were able to "enumerate" or "name" every 2-input gate?
Can we do the same for FSMs?

FSM Enumeration

These are the FSMs with 1 input and 1 output and 1 state bit. They have 8-bits in their ROM.

GOAL: List all possible FSMs in some canonical order.

- INFINITE list, but
- Every FSM has an entry in and an associated index.

i	s	o	FSM#	Truth Table
1	1	1	1	00000000
1	1	1	2	00000001
			...	...
1	1	1	256	11111111
2	2	2	257	000000...000000
2	2	2	258	000000...000001
			18,446,744,073,709,551,872 ...	...
3	3	3		000000...000000
			3.9402×10^{115} ...	
4	4	4		000000...000000

2^8 FSMs

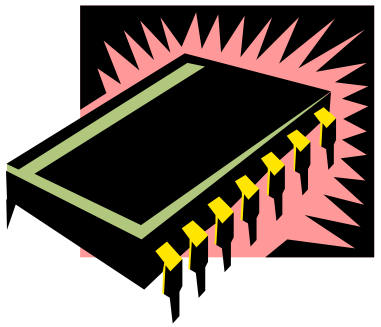
2^{64}

inputs		outputs	
i	s_N	o	s_{N+1}
0...00	0...00	10110	011
0...01			

Every possible FSM can be associated with a unique number. This requires a few wasteful simplifications. First, given an i-input, s-state-bit, and o-output FSM, we'll replace it with its equivalent n-input, n-state-bit and n-output FSM, where n is the greatest of i, s, and o. We can always ignore the extra input-bits, and set the extra output bits to 0. This allows us to discuss the i^{th} FSM

Some Perennial Favorites...

FSM_{837}	modulo 3 state machine
FSM_{1077}	4-bit counter
FSM_{1537}	Combination lock
FSM_{89143}	Cheap digital watch
$FSM_{22698469884}$	Intel Pentium CPU – rev 1
$FSM_{784362783}$	Intel Pentium CPU – rev 2
$FSM_{784363783}$	Intel Pentium II CPU



Can FSMs Compute Every Function?

Nope!

There exist many simple problems that cannot be computed by FSMs.

For instance:

Checking for balanced parenthesis

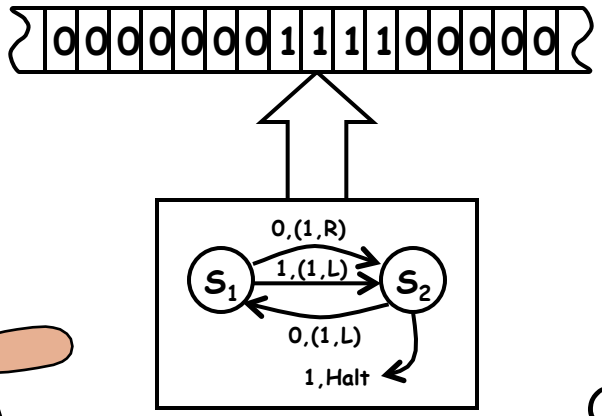
(()(()())) - Okay
(()())) - No good!

PROBLEM: Requires ARBITRARILY many states, depending on input.
Must "COUNT" unmatched LEFT parens.

But, an FSM can only keep track of a finite number of objects.

Is there a machine that can solve this problem?

Unbounded-Space Computation



Alan Turing

DURING 1920s & 1930s, much of the "science" part of computer science was being developed (long before actual electronic computers existed). Many different

"Models of Computation"

were proposed, and the classes of "functions" that each could compute were analyzed.

One of these models was the

"TURING MACHINE",

named after Alan Turing.

A Turing Machine is just an FSM which receives its inputs and writes outputs onto an infinite tape...

This simple addition solves the FSMs can only keep track of a "FINITE number of events" problem.

A Turing Machine Example

Turing Machine Specification

- Doubly-infinite tape
- Discrete symbol positions
- Finite alphabet – say {0, 1}
- Control FSM

  INPUTS:

  Current symbol on tape

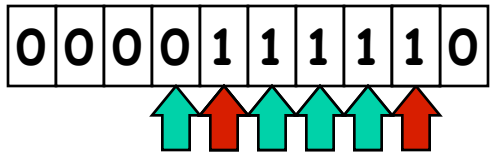
  OUTPUTS:

  write 0/1

  move Left/Right
- Initial Starting State {S0}
- Halt State {Halt}

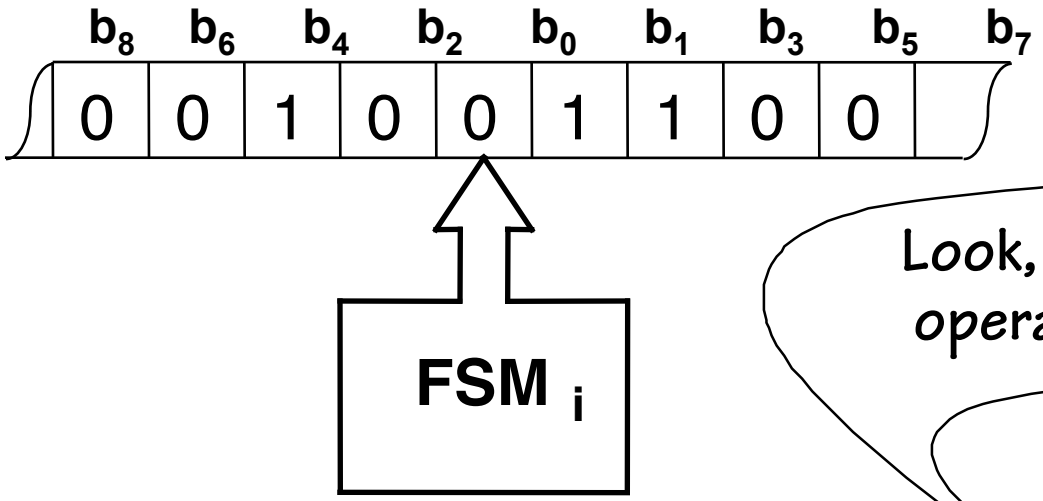
A Turing machine, like an FSM, can be specified with a truth table. The following Turing Machine implements a unary (base 1) incrementer.

Current State	Tape Input	Write Tape	Move	Next State
S0	1	1	R	S0
S0	0	1	L	S1
S1	1	1	L	S1
S1	0	0	R	Halt



Turing Machine Tapes as Integers

Canonical names for bounded tape configurations:

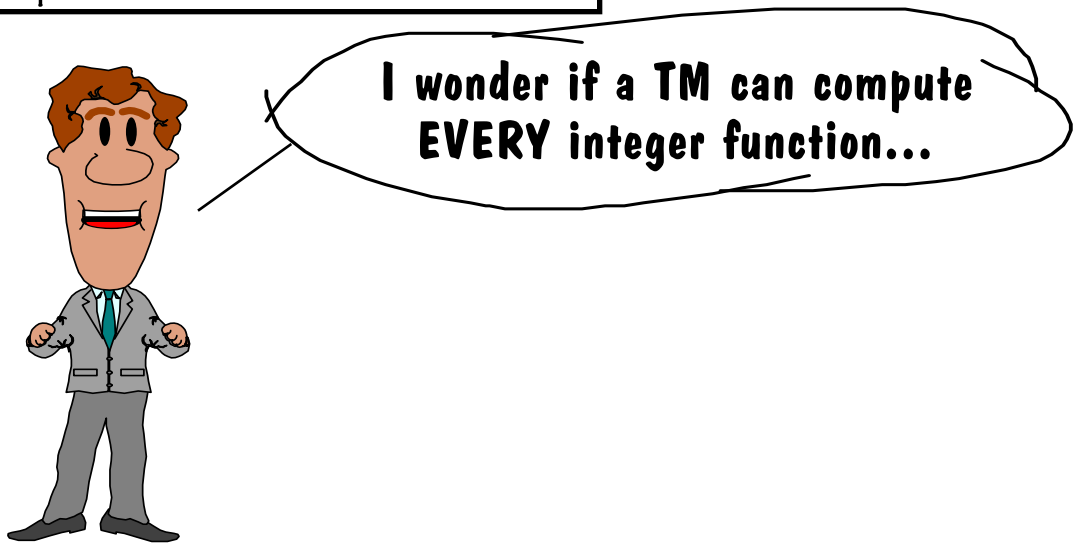


TMs as Integer Functions

Turing Machine T_i operating on Tape x,
where $x = \ldots b_8 b_7 b_6 b_5 b_4 b_3 b_2 b_1 b_0$

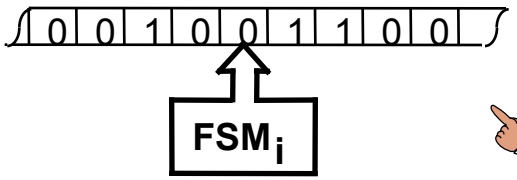
$$y = T_i[x]$$

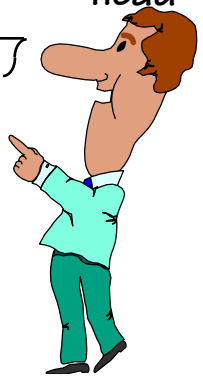
x: input tape configuration
y: output tape when TM *halts*



Alternative Models of Computation

Turing Machines [Turing]





Turing

Recursive Functions [Kleene]

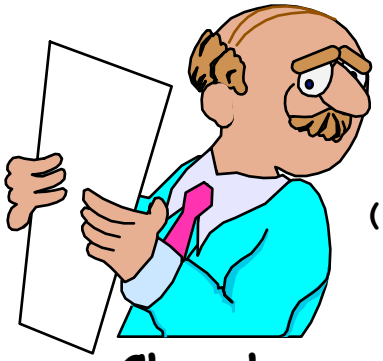
$F(0,x) \equiv x$

$F(1+y,x) \equiv 1+F(x,y)$

```
(define (fact n)
  (... (fact (- n 1)) ...))
```

Kleene

Lambda calculus [Church, Curry, Rosser…]

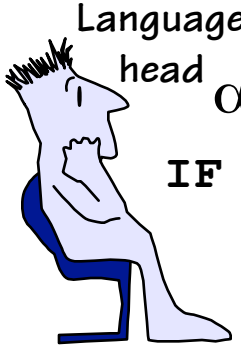


$\lambda x.\lambda y.xxy$

```
(lambda(x)(lambda(y)(x (x y))))
```

Church

Production Systems [Post, Markov]

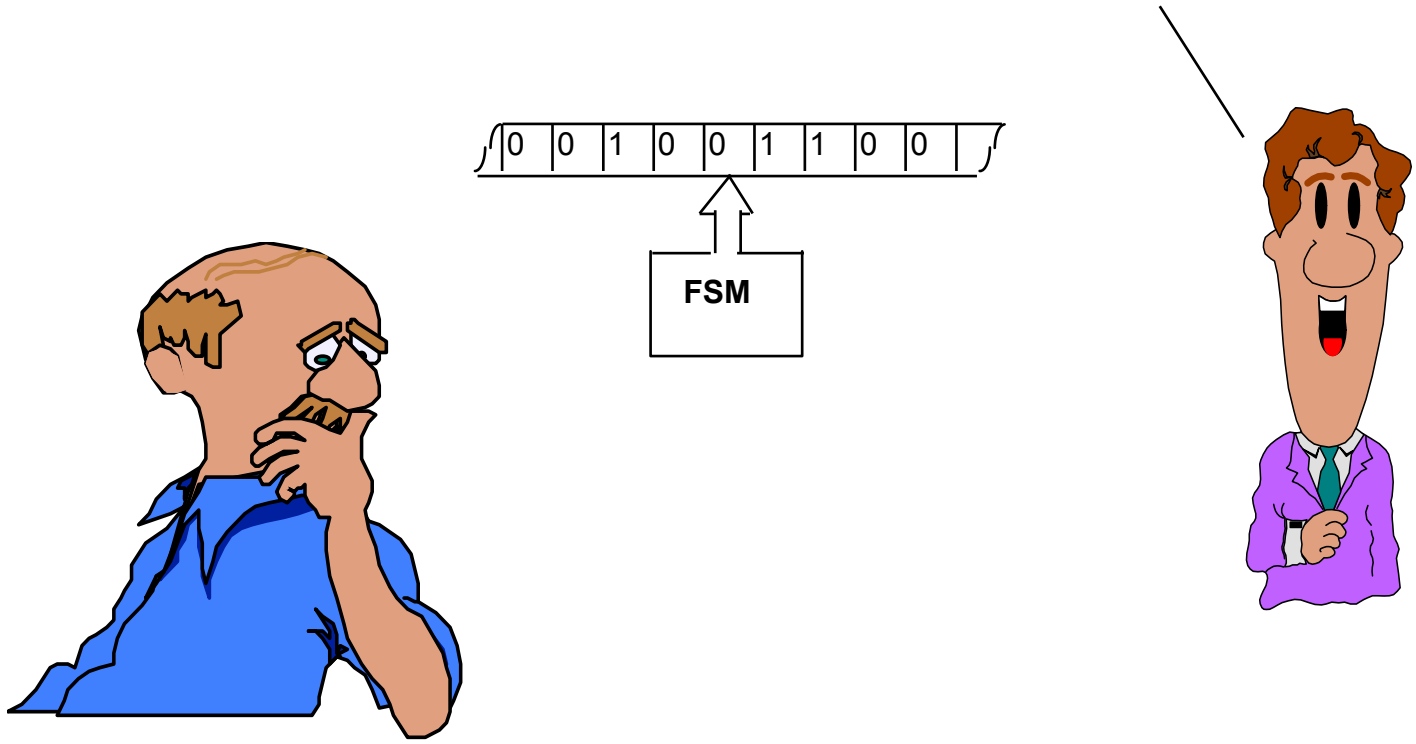


$\alpha \rightarrow \beta$

```
IF pulse=0 THEN
   patient=dead
```

Post

The 1st Computer Industry Shakeout



And the Battles Raged

Here's a Lambda Expression that does the same thing...

```
(λ(x)  .....)
```

... and here's one that computes the n^{th} root for ANY n!

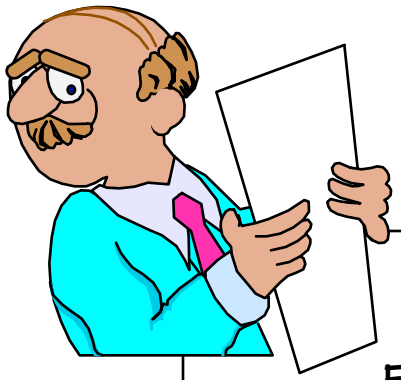
```
(λ(x n)  .....)
```

Fundamental Result: Computable Functions

Each model is capable of computing exactly the same set of integer functions!

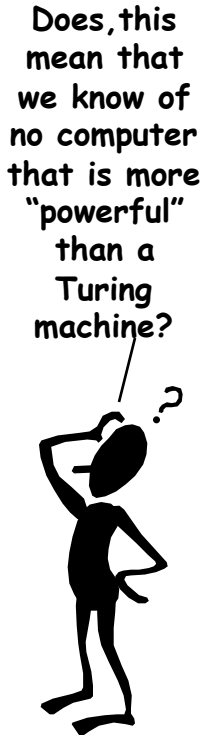
Proof Technique: Constructions that translate between models

BIG IDEA: Computability, independent of computation scheme chosen



Church's Thesis:

Every discrete function computable by ANY realizable machine is computable by some Turing machine.



Computable Functions

f(x) *computable* <=> for some k, all x:

$$f(x) = T_k[x] \equiv f_k(x)$$

Representation tricks: to compute $f_k(x,y)$

<x,y> ≡ integer whose *even* bits come from x, and whose *odd* bits come from y;
whence

$$f_k(x, y) \equiv T_k[<x, y>]$$

$f_{12345}(x,y) = x * y$

$f_{23456}(x) = 1$ iff x is prime, else 0

Enumeration of Computable functions

Conceptual table of TM behaviors...

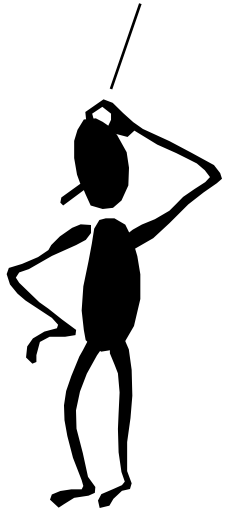
VERTICAL AXIS: Enumeration of TMs.

HORIZONTAL AXIS: Enumeration of input tapes.

(j, k) entry = result of $TM_k[j]$ -- integer, or * if never halts.

f_i	$f_i(0)$	$f_i(1)$	$f_i(2)$	•••	$f_i(j)$	•••
f_0	37 X 1	23 X 1	* X O	•••		
f_1	6 X 1	* X O	•••	•••		
•••	•••	•••	•••	•••		
f_k	•••	•••	•••	•••	$f_k(j)$	
•••	•••	•••	•••	•••	•••	

Is every Integer function that I can precisely specify computable?



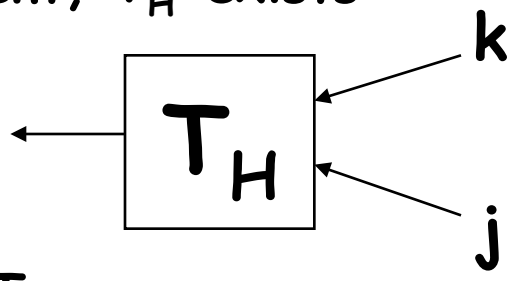
The Halting Problem: Given j, k: Does TM_k Halt with input j?

The Halting Problem

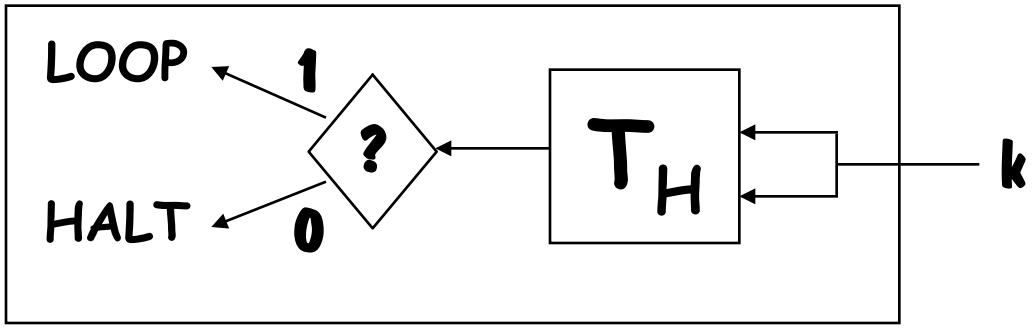
The Halting Function: $T_H[k, j] = 1$ iff $TM_k[j]$ halts, *else 0*

Can a Turing machine compute this function?

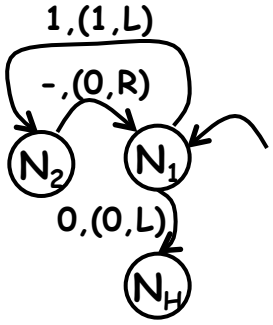
Suppose, for a moment, T_H exists:



Then we can build a T_{Nasty}:



Replace the Halt state of TH with this.



If T_H is computable then so is T_{Nasty}

$T_{Nasty}[k]$ **LOOP if** $T_k[k] = 1$ (halts)
HALT if $T_k[k] = 0$ (loops)

What does T_{Nasty}[Nasty] do?

Answer:

T_{Nasty}[Nasty] loops if T_{Nasty}[Nasty] halts

T_{Nasty}[Nasty] halts if T_{Nasty}[Nasty] loops

That's a contradiction.

Thus, T_H is uncomputable by a Turing Machine!

Net Result: There are some integer functions that Turing Machines simply cannot answer. Since, we know of no better model of computation than a Turing machine, this implies that there are some problems that defy computation.

Reality: Limits of Turing Machines

A Turing machine is formal abstraction that addresses

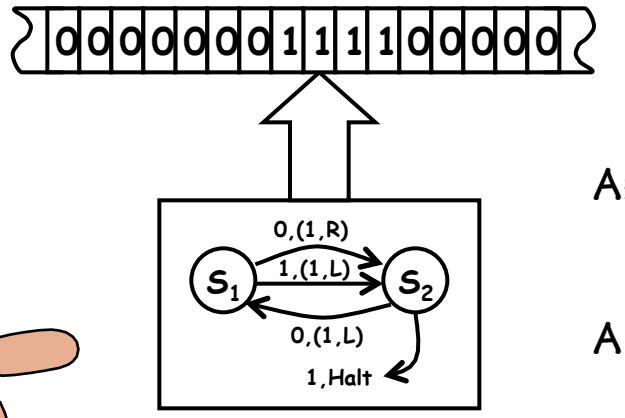
- Fundamental Limits of Computability –
  What is means to compute.
  The existence of incomputable functions.
- We know of no machine more powerful than a Turing machine in terms of the functions that it can compute.

But they ignore

- Practical coding of programs
- Performance
- Implementability
- Programmability

... these latter issues are the primary focus of contemporary computer science (Remainder of Comp 411)

Computability vs. Programmability



Alan Turing

Recall Church's thesis:

"Any discrete function computable by ANY realizable machine is computable by some Turing Machine"

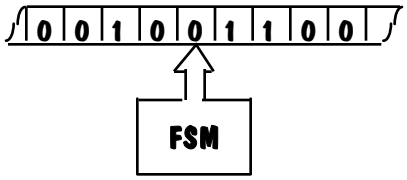
An Thusly, we've defined what it means to COMPUTE (whatever a TM can compute)

A Turing machine is nothing more that an FSM that receives inputs from, and outputs onto, an infinite tape.

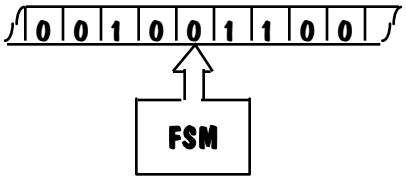
Thus far, we've been designing a new Turing machine FSM for each new function that we encounter.

Wouldn't it be nice if we could design a more general purpose computing machine?

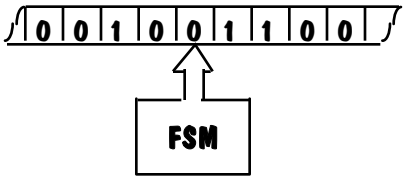
Too many Turing machines!



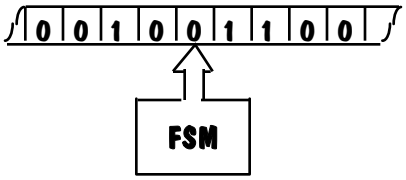
Multiplication



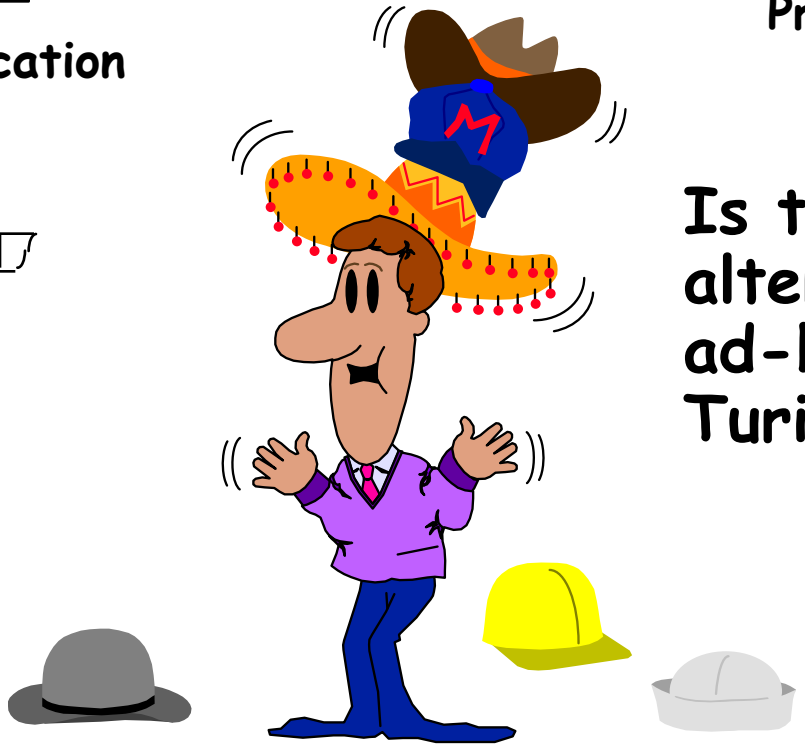
Factorization



Primality Test



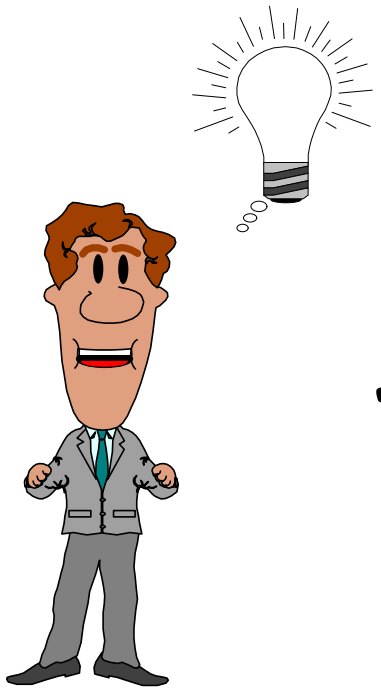
Sorting

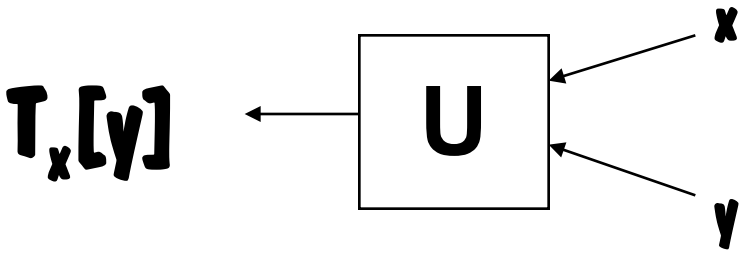


Is there an alternative to ad-hoc Turing Machines?

Programs as Data

What if we encoded the description of the FSM on our tape, and then wrote a general purpose FSM to read the tape and EMULATE the behavior of the encoded machine? Since the FSM is just a look-up table, and our machine can make reference to it as often as it likes, it seems possible that such a machine could be built.



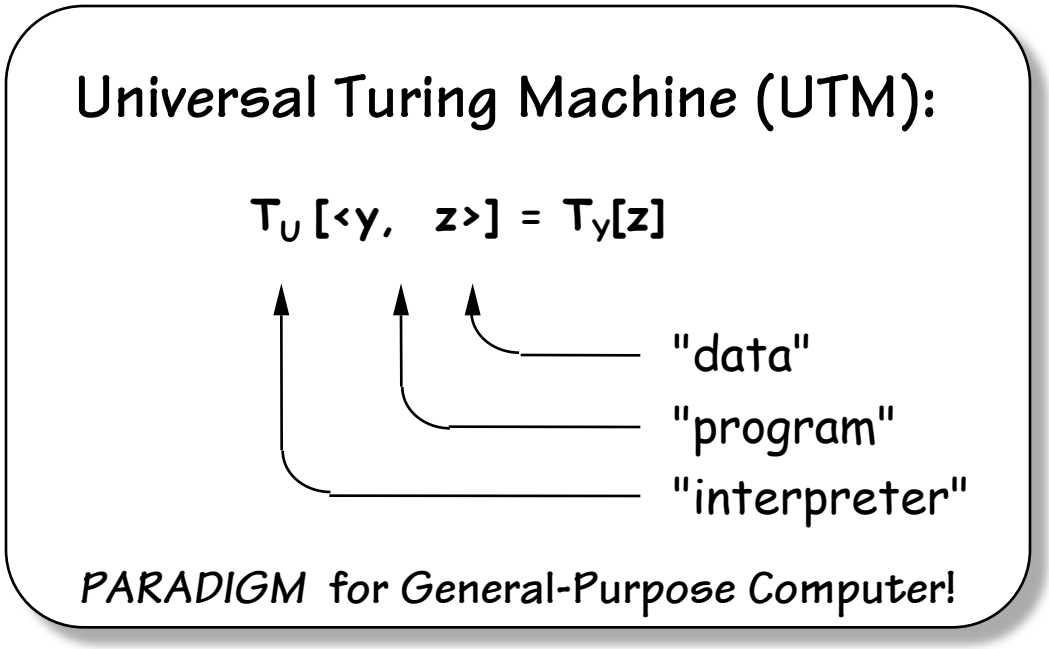


Fundamental Result: Universality

Define "Universal Function": $U(x,y) = T_x(y)$ for *every* x, y ...

Surprise! U(x,y) IS COMPUTABLE,

hence $U(x,y) = T_U(\langle x,y \rangle)$ for some U.



INFINITELY many UTMs ...

Any one of them can evaluate any computable function by simulating/ emulating/interpreting the actions of Turing machine given to it as an input.

UNIVERSALITY:

Basic requirement for a general purpose computer

Demonstrating Universality

Suppose you've *designed* Turing Machine T_K and want to show that its universal.

APPROACH:

1. **Find some *known* universal machine, say T_U.**
2. **Devise a program, P, to *simulate* T_U on T_K:**
   $T_K[<P,x>] = T_U[x]$ for all x.
3. **Since $T_U[<y,z>] = T_Y[z]$, it follows that, for all y and z.**

$$T_K\,[<P,<y,z>>] \;=\; T_U[<y,z>] \;=\; T_y[z]$$

CONCLUSION: Armed with program **P**, machine T_K can mimic the behavior of an arbitrary machine T_Y operating on an arbitrary input tape z.

HENCE T_K can compute any function that can be computed by any Turing Machine.

Interpretive Layers: What's going on?

$$T_K\,[<P,<y,z>>] \;=\; T_U[<y,z>] \;=\; T_y[z]$$

Multiple levels of interpretation:

$T_y[z]$	Application (Desired user function)
$T_U[<y,z>]$	Portable Language / Virtual Machine
$T_K[<P,<y,z>>]$	Computing Hardware / Bare Metal

Benefits of Interpretation:

BOOTSTRAP high-level functionality on very simple hardware.

Deal with ***"IDEAL"*** machines rather than real machines.

REAL MACHINES are built this way - several interpretive layers.

Power of Interpretation

BIG IDEA: Manipulate *coded representations* of computing machines, rather than the machines themselves.

- PROGRAM as a behavioral description
- SOFTWARE vs. HARDWARE
- INTERPRETER as machine which takes program and mimics behavior it describes
- LANGUAGE as interface between interpreter and program
- COMPILER as translator between languages:

INTELLECTUAL BENEFITS:

- Programs as data -- mathematical objects
- Combination, composition, generation, parameterization, etc.